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NEUTRAL EQUATIONS WITH POLYNOMIAL NONLINEARITIES ARISING IN TRANSMISSION LINES

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ABSTRACT: An existence theorem for neutral equations with polynominal nonlinearities is proved.

INTRODUCTION

It is known that in the theory of nonlinear circuits $V-I$ characteristic curve can be approximated by various nonlinear functions (cf. [1]-[3]) - polynoms, exponential functions, hyperbolic functions and their combinations. In [4]-[7] the authors in their endevour to investigate the lossless transmission lines considers the hyperbolic system

$$\frac{\partial u}{\partial x} = -L\frac{\partial i}{\partial t} \text{ и } \frac{\partial i}{\partial x} = -C\frac{\partial u}{\partial t}, \tag{1}$$

where $i = i(t,x)$, $u = u(t,x)$ are the current and the voltage respectively, while L is the series inductance, and C is the parallel capacitance per unit length of the line. The line is shorted at $x = 0$ which implies $u(0,t) = 0$ and it is connected with a nonlinear element (for instance tunnel diode) at $x = l$ (l is the length of the line), that is, $i(l,t) = f(u(l,t) + E)$, where E is the bias voltage, and $I = f(V)$ is the $V-I$ characteristic curve. In many cases $f(V)$ is third degree polynom of the type $f(V) = -aV + V^3$ (cf. [4]-[7]) or polynom of higher order [1], [2]. If the parallel capacitance C_0 is considered in the circuit the above boundary value condition becomes

$$i(l,t) = f(u(l,t) + E) + C_0\frac{\partial u(l,t)}{\partial t}.$$

Then the mixed initial-boundary problem for system (1) can be replaced by a pure initial value problem for a functional differential equation of neutral type ([4]-[7]) while in the first one obtains only functional equations. The general theory of neutral equations can be found in [8]-[11] but the polynomial nonlinearities only in particular cases are considered [4]-[7].

The main purpose of the present paper is to formulate conditions for the existence and uniqueness of solutions of neutral functional differential equations with polynomial nonlinearities in the right-hand sides in general case. Our investigations are based on fixed point approach obtained in [12]-[13].

Let us consider an initial value problem for the neutral functional differential equation of first order

$$\dot{u}(t) = F(u(\Delta_1(t)), ,..., u(\Delta_m(t), \dot{u}(\gamma_1(t),..., \dot{u}(\gamma_n(t)), t > 0,$$
$$u(t) = \varphi(t),\ \dot{u}(t) = \dot{\varphi}(t),\ t \le 0 \tag{2}$$

where $F(u_1,...,u_m,v_1,...,v_n): R^{m+n} \to R^1$,

$\varphi(t), \dot{\varphi}(t): R^1_- \to R^1$, $R^1_+ = [0,\infty), R^1_- = (-\infty,0]$,

$\Delta_i(t): R^1_+ \to R^1$ $(i = 1,2,...,m)$,

$\gamma_k(t): R^1_+ \to R^1$ $(k = 1,2,...,n)$ are prescribed functions.

Usually when one look for global solution of (2) F has to satisfy the condition of the type

$$|F(u_1,...,u_m,v_1,...,v_n)| \le a\left[\sum_{k=1}^{m}|u_k| + \sum_{s=1}^{n}|v_s|\right] \tag{3}$$

Our goal is to include in the consicleration the right-hand sides of general type:

$$F(u_1,...,u_m,v_1,...,v_n) = \sum_{s=1}^{k_1} a_s^{(1)}u_1^s + \sum_{s=1}^{k_2} a_s^{(2)}u_2^s + ...$$
$$+ \sum_{s=1}^{k_m} a_s^{(m)}u_m^s + \sum_{k=1}^{n} b_k v_k \tag{4}$$

where $a, a_s^{(l)}, b_k$ are prescribed constants.

The equations obtained in [4] - [7] are particular cases of (4):

$$C_0[\dot{u}(t)+K\dot{u}(t-h)]+\left(\frac{1}{z}-g\right)u(t)-K\left(\frac{1}{z}+g\right)u(t-h)=$$
$$=-u^3(t)-Ku^3(t-h)\ ,$$

where $C_0, K, \frac{1}{z}, g$ and h are prescribed constants.

EXISTENCE THEOREM

As usually we put $y(t)=\dot{u}(t)$ for $t>0$ and $\psi(t)=\dot{\varphi}(t)$ for $t\le 0$. Then (2) becomes assuming $y(0)=\varphi(0)=0$):

$$y(t)=F\left(\int_0^{\Delta_1(t)} y(s)ds,\dots,\int_0^{\Delta_m(t)} y(s)ds,\ y(\gamma_1(t)),\dots,\ y(\gamma_n(t))\right),t>0$$
$$y(t)=\psi(t),\ t\le 0\ . \qquad (5)$$

Indeed $y(t-t_0)=u(t)-\varphi_0-\varphi_0'(t-t_0)$ satisfies Condition s $y(0)=y(t_0-t_0)=u(t_0)-\varphi_0=0$ and

$$y'(0)=u'(t_0)-\varphi_0'=0\ (u(t_0)=\varphi_0, u'(t_0)=\varphi_0')\ .$$

Theorem 1. Let the following conditions be fulfilled:

1.1 Functions

1.2

$$\Delta_i(t),\gamma_k(t):R_+^1\to R^1\ (i=1,\dots,m;k=1,\dots,n)$$

are continuous and $\Delta_i(0)\le 0$, $\gamma_k(0)\le 0$, and

$$t-\Delta_i(t)\ge\Delta_0>0\ ,\ t-\gamma_k(t)\ge\gamma_0>0\ ,$$

where Δ_0 and $\gamma_0>0$ are constants;

1.3 the function

$$F(u_1,\dots,u_m,v_1,\dots,v_n):R^{m+n}\to R^1$$

is continuous and satisfies the conditions:

1.3.1 $\psi(t)=\dot{\varphi}(t)$ is continuous and satisfies conformity condition

1.3.2

$$\psi(0)=F\left(\int_0^{\Delta_1(0)}\psi(s)ds,\dots,\int_0^{\Delta_m(0)}\psi(s)ds,\psi(\gamma_1(0)),\dots,\psi(\gamma_n(0))\right);$$

1.2.2 $|F(u_1,\dots,u_m,v_1,\dots,v_n)|\le$

1.2.3

$$\le a_1\left(\sum_{s=0}^{k_1}|u_1|^s+\dots+\sum_{s=0}^{k_m}|u_m|^s\right)+a_2\sum_{s=1}^{n}|v_s|,$$

where a_1 and a_2 are positive constants;

1.2.3

$|F(u_1,\dots,u_m,v_1,\dots,v_n)|-|F(\tilde{u}_1,\dots,\tilde{u}_m,\tilde{v}_1,\dots,\tilde{v}_n)|\le$

$$\le b_1\left(\sum_{s=1}^{k_1}|u_1^s-\tilde{u}_1^s|+\dots+\sum_{s=1}^{k_m}|u_m^s-\tilde{u}_m^s|\right)+b_2\sum_{s=1}^{n}|v_s-\tilde{v}_s|$$

Then (5) has a unique continuous solution.

Proof. Consider the set X of all continuous functions $f(t):[0,T_0]\to R^1$ which coincide with $\psi(t)$ for $t\le 0$.

Introduce a family of pseudometrics

$$\mathscr{A}=\left\{\rho_\lambda(f,\tilde{f}):\lambda\in[0,\infty)\right\}, \text{where}$$

$$\rho_\lambda(f,\tilde{f})=\sup\left\{e^{-\lambda t}\ |\ f(t)-\tilde{f}(t)|:t\in[0,T_0]\right\}.$$

Recall that for $t\le 0$ $f(t)=\psi(t)$ and $\tilde{f}(t)=\psi(t)$.

Then the set X endowed with the family $\mathscr{A}$ becomes a uniform space $(X,\mathscr{A})$.

Introduce the set

$$M=\{f\in(X,\mathscr{A}):|\ f(t)|\le Ae^{\lambda t},t\in[0,T_0]\}\ ,$$

where A is a fixed constant which does not depend on f.

It is easy to verify that $(Tf)(t)$ is a continuous function on R^1.

First we show that the operator T defined by the right-hand side of (5) is contractive:

$$(Tf)(t)=\begin{cases}F\left(\int_0^{\Delta_1(t)}f(s)ds,\dots,\int_0^{\Delta_m(t)}f(s)ds,\ f(\gamma_1(t)),\dots,\ f(\gamma_n(t))\right),t\in[0,T_0]\\ \psi(t),\ t\le 0.\end{cases}$$

Indeed for every $f,\tilde{f}\in M$ we have $t\in[0,T_0]$ for which $\Delta_k(t)>0$:

$$|(Tf)(t)-(T\tilde{f}(t)|\le b_1\left[\sum_{s=1}^{k_1}\left|\left(\int_0^{\Delta_1(t)}f(\tau)d\tau\right)^s-\left(\int_0^{\Delta_1(t)}\tilde{f}(\tau)d\tau\right)^s\right|+\dots+\right.$$

$$\left.+\sum_{s=1}^{k_m}\left|\left(\int_0^{\Delta_m(t)}f(\tau)d\tau\right)^s-\left(\int_0^{\Delta_m(t)}\tilde{f}(\tau)d\tau\right)^s\right|\right]+b_2\sum_{s=1}^{n}|\ f(\gamma_s(t))-\tilde{f}(\gamma_s(t))|\le$$

$$\le b_1\left[\sum_{s=1}^{k_1}\left|\int_0^{\Delta_1(t)} f(\tau)d\tau-\int_0^{\Delta_1(t)}\tilde f(\tau)d\tau\right| s\left|\int_0^{\Delta_1(t)}\bar f(\tau)d\tau\right|^{s-1}+\ldots+\right.$$

$$+\sum_{s=1}^{k_m}\left|\int_0^{\Delta_m(t)} f(\tau)d\tau-\int_0^{\Delta_m(t)}\tilde f(\tau)d\tau\right| s\left|\int_0^{\Delta_m(t)}\bar f(\tau)d\tau\right|^{s-1}+$$

$$+b_2\sum_{s=1}^{n}e^{-\lambda\gamma_s(t)}\,|\,f(\gamma_s(t))-\tilde f(\gamma_s(t))\,|\,e^{\lambda\gamma_s(t)}\le$$

$$\le b_1\left[\sum_{s=1}^{k_1}sA^{s-1}\left|\int_0^{\Delta_1(t)}e^{\lambda\tau}d\tau\right|^{s-1}\left|\int_0^{\Delta_1(t)}|\,f(\tau)-\tilde f(\tau)\,|\,e^{-\lambda\tau}.e^{\lambda\tau}d\tau\right|+\ldots\right.$$

$$\left.+\sum_{s=1}^{k_m}sA^{s-1}\left|\int_0^{\Delta_m(t)}e^{\lambda\tau}d\tau\right|^{s-1}\left|\int_0^{\Delta_m(t)}|\,f(\tau)-\tilde f(\tau)\,|\,e^{-\lambda\tau}.e^{\lambda\tau}d\tau\right|\right]+$$

$$+b_2\rho_\lambda(f,\tilde f)\sum_{s=1}^{n}e^{-\lambda\gamma_s(t)}\le$$

$$\le b_1\rho_\lambda(f,\tilde f)\left[\sum_{s=1}^{k_1}sA^{s-1}\left|\int_0^{\Delta_1(t)}e^{\lambda\tau}d\tau\right|^{s}+\ldots+\sum_{s=1}^{k_m}sA^{s-1}\left|\int_0^{\Delta_m(t)}e^{\lambda\tau}d\tau\right|^{s}\right]+$$

$$+b_2\rho_\lambda(f,\tilde f)\sum_{s=1}^{n}e^{-\lambda\gamma_s(t)}\le\quad(k=\max\{k_1,k_2,\ldots,k_m\})$$

$$\le b_1\rho_\lambda(f,\tilde f)\left[\sum_{s=1}^{k_1}sA^{s-1}\left|\frac{e^{\lambda\Delta_1(t)}-1}{\lambda}\right|^{s}+\ldots+\sum_{s=1}^{k_m}sA^{s-1}\left|\frac{e^{\lambda\Delta_m(t)}-1}{\lambda}\right|^{s}\right]+$$

$$+b_2\rho_\lambda(f,\tilde f)\sum_{s=1}^{n}e^{-\lambda t+\lambda\gamma_s(t)}.e^{\lambda t}\le$$

$$\le\rho_\lambda(f,\tilde f)\left[b_1\left(\sum_{s=1}^{k_1}sA^{s-1}e^{\lambda st}\left(\frac{e^{-\lambda t+\lambda\Delta_1(t)}}{\lambda}\right)^s+\ldots\right.\right.$$

$$\left.\left.+\sum_{s=1}^{k_m}sA^{s-1}e^{\lambda st}\left(\frac{e^{-\lambda t+\lambda\Delta_m(t)}}{\lambda}\right)^s\right)+e^{\lambda t}b_2\sum_{1}^{n}e^{-\lambda\gamma_0}\right]\le(-\lambda t+\lambda\Delta_i(t)\le-\lambda\Delta_0)$$

$$\le\rho_\lambda(f,\tilde f)\left[b_1m\sum_{s=1}^{k}sA^{s-1}\frac{e^{-\lambda\Delta_0 s}}{\lambda^s}e^{\lambda st}+ne^{\lambda t}b_2e^{-\lambda\gamma_0}\right].$$

We multiply the previous inequality with $e^{-\lambda t}$ and then

$$e^{-\lambda t}\,|\,(Tf)(t)-(T\tilde f(t)\,|\le$$

$$\le\rho_\lambda(f,\tilde f)\left[b_1m\sum_{s=1}^{k}sA^{s-1}\frac{e^{-\lambda\Delta_0 s}}{\lambda^s}e^{\lambda(s-1)T_0}+nb_2e^{-\lambda\gamma_0}\right]\le$$

$$\le\rho_\lambda(f,\tilde f)\left[b_1m\frac{e^{-\lambda\Delta_0}}{\lambda}\sum_{s=1}^{k}s\left[\frac{Ae^{-\lambda\Delta_0+\lambda T_0}}{\lambda}\right]^{s-1}+nb_2e^{-\lambda\gamma_0}\right]\equiv$$

$$\equiv\rho_\lambda(f,\tilde f)B(\lambda)\quad\Rightarrow\quad\rho_\lambda(Tf,T\tilde f)\le B(\lambda)\rho_\lambda(f,\tilde f).$$

For sufficiently large λ one can see that $B(\lambda)<1$ provided $T_0-\Delta_0\le 0$. This implies that T is contractive.

We show $f\in M\ \Rightarrow\ Tf\in M$. Indeed ($\Delta_k>0$):

$$|(Tf)(t)|\le a_1\left(\sum_{s=1}^{k_1}\left|\int_0^{\Delta_1(t)}f(\tau)d\tau\right|^s+\ldots+\sum_{s=1}^{k_m}\left|\int_0^{\Delta_m(t)}f(\tau)d\tau\right|^s\right)+$$

$$+a_2\sum_{s=1}^{n}|\,f(\gamma_s(t))\,|\le$$

$$\le a_1\left(\sum_{s=1}^{k_1}A^s\left(\frac{e^{\lambda\Delta_1(t)}-1}{\lambda}\right)^s e^{\lambda st}+\ldots+\sum_{s=1}^{k_m}A^s\left(\frac{e^{\lambda\Delta_m(t)}-1}{\lambda}\right)^s e^{\lambda st}\right)+$$

$$+a_2e^{\lambda t}\sum_{s=1}^{n}e^{-\lambda t+\lambda\gamma_s(t)}A\le$$

$$\le a_1m\sum_{s=1}^{k}\left(\frac{A}{\lambda}\right)^s\left(e^{-\lambda\Delta_0+\lambda T_0}\right)^s+a_2e^{\lambda t}Ane^{-\lambda\gamma_0}\ .$$

Multiply the last inequality $e^{-\lambda t}$:

$$e^{-\lambda t}\,|\,(Tf)(t)\,|\le$$

$$\le a_1m\sum_{s=1}^{k}\left(\frac{A}{\lambda}\right)^s\left(e^{-\lambda\Delta_0+\lambda T_0}\right)^s e^{-\lambda t}+na_2Ae^{-\lambda\gamma_0}\le$$

$$\le a_1m\sum_{s=1}^{k}\left(\frac{A}{\lambda}\right)^s\left(e^{-\lambda\Delta_0+\lambda T_0}\right)^s+na_2Ae^{-\lambda\gamma_0}\le A,$$

which is always satisfied for sufficiently large λ and $T_0-\Delta_0\le 0$.

In order to be satified all conditions of the fixed point theorem [12] we have to find such an element x_0 that

$$\rho_{j^k(\lambda)}(x_0,Tx_0)\le Q<\infty\ (k=1,2,\ldots)$$

Here the map $j:A\to A$ (cf. [12]) is $j(\lambda)=\lambda$. One can choose x_0

$$x_0(t)=\begin{cases}\psi(0), & t\ge 0\\ \psi(t), & t\le 0\end{cases}.$$

Then we obtain

$$\rho_{j^k(\lambda)}(x_0, Tx_0) =$$

$$= \sup\{|\psi(0) - F(\Delta_1(t)\psi(0),...,\Delta_m(t)\psi(0), \psi(0),...,\psi(0))| e^{-\lambda t} : t \in [0, T_0]\} < \infty$$

The last supremum exists because

$|\Delta_i(t)| \le |t - \Delta_0|$ and consequently

$$e^{-\lambda t}\sum_{s=1}^{k_i}(|\Delta_i(t)|\psi(0))^s < \infty\,,\ (i = 1,2,..., m)\,.$$

Therefore T has a unique fixed point [12], which is a solution of (5).

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