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ON *j*-LIPSCHITZIAN MAPPINGS IN PSEUDOMETRICALLY CONVEX UNIFORM SPACES

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SUMMARY
Conditions guaranteeing global *j*-Lipschitzicity of mappings defined in complete pseudometrically convex uniform spaces are given. Such mappings arise in many applications mentioned in the references.

Let $(X, \mathbf{A})$ be a complete T_2-separated uniform space whose uniformity is generated by a saturated family of pseudometrics $\mathbf{A} = \{d_a(x,y): a \in A\}$, where A is an index set. Basic notions concerning the uniform spaces can be found in (Weil, 1938; Kelley, 1959; Isbell, 1964; Page, 1978). A uniform space $(X, \mathbf{A})$ is said to be pseudometrically convex if for every $x, y \in X$ there is $z \in X (z \neq x \neq y)$ such that

$$d_a(x,y) = d_a(x,z) + d_a(z,y)$$

for every $a \in A$ provided $d_a(x,y) > 0$ (cf. Angelov et al., 1997). The metrical segment between *x* and *y* with respect to the pseudometric $d_a(.,.)$ we denote by $[x,y]_a$, $a \in A$ and

$$(x,y)_a = [x,y]_a \setminus \{x,y\}.$$

Let $j: A \to A$ be a map of the index set into itself whose iterates are defined inductively as follows

$$j^0(a) = a, j^k(a) = j(j^{k-1}(a)) \quad (k \in N).$$

Further on by $(X, \mathbf{A})$ and $(Y, \mathbf{A})$ we mean uniform spaces whose index sets for their families of pseudometrics coincide, i.e. we have

$$(X, \{d_a(x,y): a \in A\}), (Y, \{d_a(x,y): a \in A\}).$$

Let $\{M_a\}_{a \in A}$ be a family of positive constants where A is the same index set. A mapping $T: X \to Y$ is called *j*-Lipschitzian if for every $x, y \in X$ is satisfied

$$d_a(Tx, Ty) \leq M_a d_{j(a)}(x,y) \text{ for } a \in A.$$

The main goal of the present note is to formulate local conditions which imply the *j*-Lipschitzian character of mappings defined in complete pseudometrically convex uniform spaces. Such mappings are generated by functional differential equations treated in (Angelov, 1987; 1989, 2000).

Prior to formulate the main result we present an example of pseudometrically convex uniform space. Consider the set $C(R^1)$ of all continuous functions $f: R^1 \to R^1$. Let us fix an arbitrary function $b(.) \in C(R^1)$ and assume that $b(t)$ is unbounded and positive on R^1. Define the set $C_b(R^1) = \{f \in C(R^1): |f(t)| \leq b(t)\}$. It is easy to see that the sum of two functions from $C_b(R^1)$ does not belong to $C_b(R^1)$ in general case. This means $C_b(R^1)$ is neither a Banach space nor linear topological space. Denote by A the family of all compact intervals $a = [p,q] \subset R^1$ and introduce a family of pseudometrics $\mathbf{A} = \{d_a(f,g): a \in A\}$ on $C_b(R^1)$ where

$$d_a(f,g) = \| f - g \|_a, \| f \|_a = \sup\{|f(t)|: t \in a\}.$$

For every $f, g \in C_b(R^1)$ with $f \neq g$ we define the functions $h_\lambda = (1-\lambda) f(t) + \lambda g(t)$, where the parameter $\lambda \in [0,1]$. For each fixed $a \in A$ we have

$$d_a(f, h_\lambda) = \| f - (1-\lambda) f - \lambda g \|_a = \lambda d_a(f,g)$$

$$d_a(g, h_\lambda) = \| g - (1-\lambda) f - \lambda g \|_a = (1-\lambda) d_a(f,g).$$

Obviously $d_a(f,g) = d_a(f,h_\lambda) + d_a(h_\lambda,g)$. So by $[f,g]_a$ we denote the closed metrical segment $[f,g]_a = \{h_\lambda : \lambda \in [0,1]\}$, while by $(f,g)_a = \{h_\lambda : \lambda \in (0,1)\}$. It is easy to verify that $d_a(f,g) > 0$ implies $d_a(f,h_\lambda) > 0$ and $d_a(g,h_\lambda) > 0$ for $\lambda \in (0,1)$ and vice versa.

The mapping $T : X \to Y$ is said to be almost directionally *j*-Lipschitzian on X if for each $x, y \in X$ with $x \neq y$ the following inequality holds:

$$\inf\left[\frac{d_a(Tx,Tz)}{d_{j(a)}(x,z)} : z \in (x,y)_{j(a)}\right] \le M_a \qquad (1)$$

for $d_{j(a)}(x,y) > 0$ and $d_a(Tx,Ty) = 0$ for $d_{j(a)}(x,y) = 0$.

The above definition extends the corresponding notion in metric spaces (cf. Kirk et al., 1977).

For $d_{j(a)}(x,y) = 0$ there is no metrical segment between x and y with respect to $j(a)$.

Recall that a mapping $T : X \to Y$ is closed if for $\{x_n\} \subset X$ the conditions $\lim_n x_n = x$ and $\lim_n T(x_n) = y$ imply $x \in X$ and $Tx = y$.

THEOREM. Let $(X,\mathbf{A})$ and $(Y,\mathbf{A})$ be complete T_2-separated uniform spaces with $(X,\mathbf{A})$ being pseudometrically convex. Let $T : (X,\mathbf{A}) \to (Y,\mathbf{A})$ be almost directionally *j*-Lipschitzian mapping with a family of positive constants $\{M_a\}_{a\in A}$. If T is closed, then T is *j*-Lipschitzian mapping on the whole $(X,\mathbf{A})$.

Proof: Let x and y be two distinct elements of X, i.e. $x \neq y$. Let us choose a new family of positive constants $\{M'_a\}_{a\in A}$ such that $M'_a > M_a$ for every fixed $a \in A$. By Ω we denote the countable ordinals and fixed $\gamma \in \Omega$ (cf. Ch.XV, Angelov, 1987). For all $\alpha \in \Omega$ less than γ define the set $\{x_\alpha\}$ so that

(γ1) $x_0 = x$;

(γ2) if $x_\alpha = y$ for some α and $\alpha \le \eta < \gamma$, then $x_\eta = y$;

(γ3) if $\alpha < \beta < \eta < \gamma$ and $x_\eta \neq y$, then $x_\beta \in (x_\alpha, x_\eta)_{j(a)}$ for those $a \in A$ for which $d_{j(a)}(x_\alpha, x_\eta) > 0$;

(γ4) If $\beta < \eta < \gamma$ and $x_\eta \neq y$, then $x_\eta \in (x_\beta, y)_{j(a)}$;

(γ5) T is *j*-Lipschitzian with a family $\{M'_a\}_{a\in A}$ on the set $\{x_\eta : \eta < \gamma\}$.

If γ has a predecessor $\mu \in \Omega$, that is, $\gamma = \mu + 1$, then we put $x_\gamma = y$.

If $x_\mu \neq y$ and $d_{j(a)}(x_\mu, y) > 0$ in view of the definition (1) we choose $x_\gamma \in (x_\mu, y)_{j(a)}$ so that

$$d_a(Tx_\gamma, Tx_\mu) \le M'_a d_{j(a)}(x_\gamma, x_\mu)$$

(recall that $d_{j(a)}(x_\mu, y) > 0$ implies $d_{j(a)}(x_\mu, x_\gamma) > 0$).

In what follows we show that (γ1)- (γ3) hold for $\eta = \gamma$. If $\alpha < \mu$, then (γ4) implies

$$\begin{aligned} d_{j(a)}(x_\alpha, x_\gamma) &\le d_{j(a)}(x_\alpha, x_\mu) + d_{j(a)}(x_\mu, x_\gamma) = \\ &= d_{j(a)}(x_\alpha, y) - d_{j(a)}(x_\mu, y) + d_{j(a)}(x_\mu, y) - d_{j(a)}(x_\gamma, y) = \\ &= d_{j(a)}(x_\alpha, y) - d_{j(a)}(x_\gamma, y) \le d_{j(a)}(x_\alpha, x_\gamma), \end{aligned}$$

that is, $d_{j(a)}(x_\alpha, x_\gamma) = d_{j(a)}(x_\alpha, x_\mu) + d_{j(a)}(x_\mu, x_\gamma)$ or $x_\mu \in (x_\alpha, x_\gamma)_{j(a)}$. If $\alpha < \beta < \mu < \gamma$, then (γ3) implies

$$\begin{aligned} d_{j(a)}(x_\alpha, x_\gamma) &= d_{j(a)}(x_\alpha, x_\mu) + d_{j(a)}(x_\mu, x_\gamma) = \\ &= d_{j(a)}(x_\alpha, x_\beta) + d_{j(a)}(x_\beta, x_\mu) + d_{j(a)}(x_\mu, x_\gamma) \ge \\ &\ge d_{j(a)}(x_\alpha, x_\beta) + d_{j(a)}(x_\beta, x_\gamma) \ge d_{j(a)}(x_\alpha, x_\gamma), \end{aligned}$$

that is $d_{j(a)}(x_\alpha, x_\gamma) = d_{j(a)}(x_\alpha, x_\beta) + d_{j(a)}(x_\beta, x_\gamma)$ which means $x_\beta \in (x_\alpha, x_\gamma)_{j(a)}$. So (γ3) is thus proved for $\eta = \gamma$.

For $\beta < \gamma$ we have

$$\begin{aligned} d_{j(a)}(x_\beta, y) &= d_{j(a)}(x_\beta, x_\mu) + d_{j(a)}(x_\mu, y) = \\ &= d_{j(a)}(x_\beta, x_\mu) + d_{j(a)}(x_\mu, x_\gamma) + d_{j(a)}(x_\gamma, y). \\ &= d_{j(a)}(x_\beta, x_\gamma) + d_{j(a)}(x_\gamma, y) \end{aligned}$$

Therefore (γ4) holds for $\eta = \gamma$.

We have to show that (γ5) holds for $\eta = \gamma$.

Suppose $\beta < \gamma$. As we have already shown $x_\beta \in (x_\alpha, x_\gamma)_{j(a)}$ and

$$\begin{aligned} d_a(Tx_\beta, Tx_\gamma) &\le d_a(Tx_\beta, Tx_\mu) + d_a(Tx_\mu, Tx_\gamma) \le \\ &\le M'_a d_{j(a)}(x_\beta, x_\mu) + M'_a d_{j(a)}(x_\mu, x_\gamma) = M'_a d_{j(a)}(x_\beta, x_\gamma) \end{aligned}$$

which proves (γ5).

If $\gamma \in \Omega$ is a limit ordinal, then we can choose a increasing sequence of ordinals $\{\gamma_n\}_{n=1}^\infty$, $\gamma_n \in \Omega$ such that $\lim_n \gamma_n = \gamma$. In view of (γ1) and (γ3)

$$d_{j(a)}(x_0, x_{\gamma_{n+1}}) = d_{j(a)}(x_0, x_{\gamma_n}) + d_{j(a)}(x_{\gamma_n}, x_{\gamma_{n+1}})$$

which implies $d_{j(a)}(x_0, x_{\gamma_{n+1}}) \ge d_{j(a)}(x_0, x_{\gamma_n})$, that is, the sequence $\{d_{j(a)}(x_0, x_{\gamma_n})\}_{n=1}^{\infty}$ is non-decreasing. Since (γ4) implies

$$d_{j(a)}(x_0, y) = d_{j(a)}(x_0, x_{\gamma_n}) + d_{j(a)}(x_{\gamma_n}, y),$$
$$d_{j(a)}(x_0, x_{\gamma_n}) = d_{j(a)}(x_0, y) - d_{j(a)}(x_{\gamma_n}, y)$$
$$\text{or } d_{j(a)}(x, x_{\gamma_n}) \le d_{j(a)}(x, y)$$

which implies the convergence of the sequence $\{d_{j(a)}(x_0, x_{\gamma_n}\}_{n=1}^{\infty}$. If we put $x_{\gamma_0} = x$ then

$$d_{j(a)}(x, x_{\gamma_n}) = \sum_{k=0}^{n-1} d_{j(a)}(x_{\gamma_k}, x_{\gamma_{k+1}}),$$

which implies

$$\sum_{k=0}^{\infty} d_{j(a)}(x_{\gamma_k}, x_{\gamma_{k+1}}) \le d_{j(a)}(x, y) < \infty .$$

This means that $\{x_{\gamma_n}\}_{n=0}^{\infty}$ is a Cauchy sequence in $(X, \mathbf{A})$. The completeness of X implies an existence of an element $z \in X$ such that $\lim_n x_{\gamma_n} = z$. We can define $x_{\gamma} = z$.

If for any $\alpha < \gamma$ we have $x_{\alpha} = y$, then $\{x_{\gamma_n}\}$ is eventually the constant sequence {*y*}. In this case (γ2) - (γ5) are satisfied for $\eta = \gamma$.

Since $\gamma_n < \gamma$ then (γ5) implies

$$d_a(Tx_{\gamma_n}, Tx_{\gamma_m}) \le M'_a d_{j(a)}(x_{\gamma_n}, x_{\gamma_m})$$

for all m and n. We can assume without loss of generality that $d_{j(a)}(x_m, x_n) > 0$. Otherwise we remove the elements for which $d_{j(a)}(x_m, x_n) = 0$ and renumber the rest ones. Consequently $\{Tx_{\gamma_n}\}_n$ is a Cauchy sequence in Y. But Y is a complete uniform space and T has a closed graph. Then $\lim_n Tx_{\gamma_n} = Tx_{\gamma}$. Let $\alpha < \beta < \gamma$ and let n be chosen sufficiently large such that $\gamma_n \ge \beta$. In view of (γ3) we have

$$d_{j(a)}(x_{\alpha}, x_{\gamma_n}) = d_{j(a)}(x_{\alpha}, x_{\beta}) + d_{j(a)}(x_{\beta}, x_{\gamma_n}).$$

Passing to the limit $n \to \infty$ in the last equality we obtain

$$d_{j(a)}(x_{\alpha}, x_{\gamma}) = d_{j(a)}(x_{\alpha}, x_{\beta}) + d_{j(a)}(x_{\beta}, x_{\gamma}),$$

that is, $x_{\beta} \in (x_{\alpha}, x_{\gamma})_{j(a)}$. Also (γ4) implies

$$d_{j(a)}(x_{\beta}, y) = d_{j(a)}(x_{\beta}, x_{\gamma_n}) + d_{j(a)}(x_{\gamma_n}, y)$$

and after $n \to \infty$ we obtain $x_{\gamma} \in (x_{\beta}, y)_{j(a)}$. Consequently (γ3) and (γ4) are satisfied for $\eta = \gamma$. Condition (γ5) implies

$$d_a(Tx_{\beta}, Tx_{\gamma_n}) \le M'_a d_{j(a)}(x_{\beta}, x_{\gamma_n})$$

(provided $d_{j(a)}(x_{\beta}, x_{\gamma_n}) > 0$) hence by $n \to \infty$

$$d_a(Tx_{\beta}, Tx_{\gamma}) \le M'_a d_{j(a)}(x_{\beta}, x_{\gamma}).$$

Finally we obtained a set $\{x_{\gamma} : \gamma \in \Omega\}$ in X so that (γ1) - (γ5) are satisfied. If $x_{\gamma} \ne y$ for all $\gamma \in \Omega$ then (γ3) implies that the set $Z = \{d_{j(a)}(x, x_{\gamma}) : \gamma \in \Omega\}$ is a discrete set of real numbers. In view of Theorem 2, Ch. XV, (Sierpinski, 1965) Z is non-denumerable set. The obtained contradiction implies that for some $\gamma \in \Omega$, $x_{\gamma} = y$ and then (γ5) implies

$$d_a(Tx, Ty) \le M'_a d_{j(a)}(x, y).$$

The last inequality is valid for arbitrary $M'_a > M_a$ and consequently

$$d_a(Tx, Ty) \le M_a d_{j(a)}(x, y).$$

Theorem is thus proved.

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