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ON THE EQUIVALENCE OF DIFFERENTIAL SYSTEMS ARISING IN ELECTROMAGNETIC TWO-BODY PROBLEM

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SUMMARY
The equivalence of two systems of equations of motion arising in electromagnetic two-body problem is obtained.

In the present note we consider two systems of equations of motion arising in electromagnetic two-body problem (Synge, 1940; Synge, 1960) and formulated in (Angelov, 2002).

First we recall some denotations and results from (Angelov, 2002; Angelov, 2000) conserning

J. L. Synge's equations of motion
As in (Synge, 1940) we denote by

$$x^{(p)}=(x_1^{(p)}(t),x_2^{(p)}(t),x_3^{(p)}(t),x_4^{(p)}(t)=ict)$$

$(p=1,2,\ i^2=-1)$ the space-time coordinates of the moving particles, by m_p - their proper massses, by e_p - their charges, c - the speed of the light. The coordinates of the velocity vectors are

$$u^{(p)}=(u_1^{(p)}(t),u_2^{(p)}(t),u_3^{(p)}(t))\quad(p=1,2).$$

The coordinates of the unit tangent vectors to the world-lines are

$$\lambda_\alpha^{(p)}=\frac{\gamma_p u_\alpha^{(p)}(t)}{c}=\frac{u_\alpha^{(p)}(t)}{\Delta_p}\ (\alpha=1,2,3),\ \lambda_4^{(p)}=i\gamma_p=\frac{ic}{\Delta_p},$$

where

$$\gamma_p=\left(1-\frac{1}{c^2}\sum_{\alpha=1}^{3}[u_\alpha^{(p)}(t)]^2\right)^{-\frac{1}{2}},\ \Delta_p=\left(c^2-\sum_{\alpha=1}^{3}[u_\alpha^{(p)}(t)]^2\right)^{\frac{1}{2}}.$$

It follows $\gamma_p=c/\Delta_p$.

By $<.,.>_4$ we denote the scalar product in the Minkowski space, while by $<.,.>$ - the scalar product in 3-dimensional Euclidean subspace. Synge's equations of motion modeling the interaction of two moving charged particles are the following:

$$m_p\frac{d\lambda_r^{(p)}}{ds_p}=\frac{e_p}{c^2}F_{rn}^{(p)}\lambda_n^{(p)}\quad(r=1,2,3,4)\qquad(1)$$

where the elements of proper time are

$$ds_p=\frac{c}{\gamma_p}dt=\Delta_p dt\ (p=1,2).$$

Recall that in (1) there is a summation in $n\ (n=1,2,3)$.

The elements $F_{rn}^{(p)}$ of the electromagnetic tensors are derived by the retarded Lienard-Wiecherd potentials

$$A_r^{(p)}=-\frac{e_p\lambda_r^{(p)}}{\left\langle\lambda^{(p)},\xi^{(pq)}\right\rangle_4}\quad(r=1,2,3,4)\text{, that is}$$

$$F_{rn}^{(p)}=\frac{\partial A_n^{(p)}}{\partial x_r^{(p)}}-\frac{\partial A_r^{(p)}}{\partial x_n^{(p)}}\quad\xi^{(pq)}.$$

By $\xi^{(pq)}$ we denote the isotropic vectors (cf. Synge, 1940; 1960) drawn into the past:

$$\xi^{(pq)}=(x_1^{(p)}(t)-x_1^{(q)}(t-\tau_{pq}(t)),$$
$$x_2^{(p)}(t)-x_2^{(q)}(t-\tau_{pq}(t)),x_3^{(p)}(t)-x_3^{(q)}(t-\tau_{pq}(t)),ic\tau_{pq}(t))$$

where $\left\langle \xi^{(p,q)}, \xi^{(p,q)} \right\rangle_4 = 0$ or

$$\tau_{pq}(t) = \frac{1}{c}\left(\sum_{\beta=1}^{3} [x_\beta^{(p)}(t) - x_\beta^{(q)}(t - \tau_{pq}(t))]^2 \right)^{\frac{1}{2}} \qquad (*)$$

$((pq) = (12), (21))$.

Calculating $F_{rn}^{(p)}$ as in (Angelov, 1990) we write equations from (2) in the form:

$$\frac{d\lambda_\alpha^{(p)}}{ds_p} = \frac{Q_p}{c^2}\left\{ \frac{\xi_\alpha^{(pq)}\left\langle \lambda^{(p)}, \lambda^{(q)} \right\rangle_4 - \lambda_\alpha^{(q)}\left\langle \lambda^{(p)}, \xi^{(pq)} \right\rangle_4}{\left\langle \lambda^{(q)}, \xi^{(pq)} \right\rangle_4^3} \left[1 + \left\langle \xi^{(pq)}, \frac{d\lambda^{(q)}}{ds_q} \right\rangle_4 \right] + \frac{1}{\left\langle \lambda^{(q)}, \xi^{(pq)} \right\rangle_4^2} \left[\left\langle \lambda^{(p)}, \xi^{(pq)} \right\rangle_4 \frac{d\lambda^{(q)}}{ds_q} - \left\langle \xi^{(pq)}, \frac{d\lambda^{(q)}}{ds_q} \right\rangle_4 \xi_\alpha^{(pq)} \right] \right\} \quad (\alpha = 1,2,3) \qquad (2.\alpha)$$

$$\frac{d\lambda_4^{(p)}}{ds_p} = \frac{Q_p}{c^2}\left\{ \frac{\xi_4^{(pq)}\left\langle \lambda^{(p)}, \lambda^{(q)} \right\rangle_4 - \lambda_4^{(q)}\left\langle \lambda^{(p)}, \xi^{(pq)} \right\rangle_4}{\left\langle \lambda^{(q)}, \xi^{(pq)} \right\rangle_4^3} \left[1 + \left\langle \xi^{(pq)}, \frac{d\lambda^{(q)}}{ds_q} \right\rangle_4 \right] + \frac{1}{\left\langle \lambda^{(q)}, \xi^{(pq)} \right\rangle_4^2} \left[\left\langle \lambda^{(p)}, \xi^{(pq)} \right\rangle_4 \frac{d\lambda^{(q)}}{ds_q} - \left\langle \xi^{(pq)}, \frac{d\lambda^{(q)}}{ds_q} \right\rangle_4 \xi_4^{(pq)} \right] \right\} \qquad (2.4)$$

where $Q_p = \frac{e_1.e_2}{m_p}$, $(p = 1,2)$. Further on we denote

$u^{(q)} \equiv u^{(q)}(t - \tau_{pq})$,

$$\lambda^{(q)} = (\gamma_{pq} u_1^{(q)} / c, \gamma_{pq} u_2^{(q)} / c, \gamma_{pq} u_3^{(q)} / c, i\gamma_{pq}) = = (u_1^{(q)} / \Delta_{pq}, u_2^{(q)} / \Delta_{pq}, u_3^{(q)} / \Delta_{pq}, ic / \Delta_{pq}),$$

where $\gamma_{pq} = \left(1 - \frac{1}{c^2} \sum_{\alpha=1}^{3} [u_\alpha^{(q)}(t - \tau_{pq}(t))]^2 \right)^{-\frac{1}{2}}$,

$\Delta_{pq} = \left(c^2 - \sum_{\alpha=1}^{3} [u_\alpha^{(q)}(t - \tau_{pq}(t))]^2 \right)^{\frac{1}{2}}$ and

$$\frac{d\lambda_\alpha^{(p)}}{ds_p} = \frac{d\left(\frac{\gamma_p}{c} u_\alpha^{(p)} \right)}{\frac{c}{\gamma_p} dt} = \frac{d\left(\frac{u_\alpha^{(p)}}{\Delta_p dt} \right)}{\Delta_p dt} = = \frac{1}{\Delta_p^2} \dot{u}_\alpha^{(p)} + \frac{u_\alpha^{(p)}}{\Delta_p^4} \left\langle u^{(p)}, \dot{u}^{(p)} \right\rangle \quad (\alpha = 1,2,3)$$

$$\frac{d\lambda_4^{(p)}}{ds_p} = \frac{d(i\gamma_p)}{\frac{c}{\gamma_p} dt} = \frac{icd\left(\frac{1}{\Delta_p} \right)}{\Delta_p dt} = \frac{ic}{\Delta_p^4} \left\langle u^{(p)}, \dot{u}^{(p)} \right\rangle,$$

where the dot means a differentiation in *t*.

Proceeding as in (Angelov, 2000) and proving that 4-th and 8-th equations are a consequence of the rest ones we obtain a system of 6 equations. Now we are able to formulate the initial value problem for the above equations in the following way: to find unknown velocities $u_\alpha^{(p)}(t)$ $(p = 1,2; \alpha = 1,2,3)$, for $t \geq 0$ satisfying equations $(3_{1\alpha})$, $(3_{2\alpha})$ of motion (written in details below):

$$\frac{1}{\Delta_1} \dot{u}_\alpha^{(1)} + \frac{u_\alpha^{(1)}}{\Delta_1^3} \left\langle u^{(1)}, \dot{u}^{(1)} \right\rangle = = \frac{Q_1}{c^2} \left\{ \frac{\left[c^2 - \left\langle u^{(1)}, u^{(2)} \right\rangle \right] \xi_\alpha^{(12)} - \left[c^2 \tau_{12} - \left\langle u^{(1)}, \xi^{(12)} \right\rangle \right] u_\alpha^{(2)}}{\left[c^2 \tau_{12} - \left\langle u^{(1)}, \xi^{(12)} \right\rangle \right]^3} \times \right.$$
$$\times \frac{\Delta_{12}^4 + D_{12} \Delta_{12}^2 \left\langle \xi^{(12)}, \dot{u}^{(2)} \right\rangle + \left(\left\langle \xi^{(12)}, u^{(2)} \right\rangle - c^2 \tau_{12} \right) \left\langle u^{(2)}, \dot{u}^{(2)} \right\rangle}{\Delta_{12}^2} +$$
$$+ D_{12} \frac{\left(\left\langle u^{(1`)}, \xi^{(12)} \right\rangle - c^2 \tau_{12} \right) \dot{u}_\alpha^{(2)} - \left\langle u^{(1)}, \dot{u}^{(2)} \right\rangle \xi_\alpha^{12}}{\left[ c^2 \tau_{12} - \left\langle u^{(2`)}, \xi^{(12)} \right\rangle \right]^2} + \qquad (3_{1\alpha})$$
$$+ \frac{D_{12}}{\Delta_{12}^2} \cdot \frac{\left(\left\langle u^{(1`)}, \xi^{(12)} \right\rangle - c^2 \tau_{12} \right) u_\alpha^{(2)} \left\langle u^{(2)}, \dot{u}^{(2)} \right\rangle}{\left[ c^2 \tau_{12} - \left\langle u^{(2`)}, \xi^{(12)} \right\rangle \right]^2} +$$
$$\left. + \frac{D_{12}}{\Delta_{12}^2} \cdot \frac{\left(c^2 - \left\langle u^{(1`)}, u^{(2)} \right\rangle \right) \xi_\alpha^{(12)} \left\langle u^{(2)}, \dot{u}^{(2)} \right\rangle}{\left[ c^2 \tau_{12} - \left\langle u^{(2`)}, \xi^{(12)} \right\rangle \right]^2} \right\}.$$

$$\frac{1}{\Delta_2}\dot{u}_\alpha^{(2)}+\frac{u_\alpha^{(2)}}{\Delta_1^3}\left\langle u^{(2)},\dot{u}^{(2)}\right\rangle=$$
$$=\frac{Q_1}{c^2}\left\{\frac{\left[c^2-\left\langle u^{(1)},u^{(2)}\right\rangle\right]\xi_\alpha^{(21)}-\left[c^2\tau_{12}-\left\langle u^{(1)},\xi^{(21)}\right\rangle\right]u_\alpha^{(1)}}{\left[c^2\tau_{12}-\left\langle u^{(1)},\xi^{(12)}\right\rangle\right]^3}\times\right.$$
$$\times\frac{\Delta_{21}^4+D_{21}\Delta_{21}^2\left\langle\xi^{(21)},\dot{u}^{(1)}\right\rangle+\left(\left\langle\xi^{(21)},u^{(1)}\right\rangle-c^2\tau_{21}\right)\left\langle u^{(1)},\dot{u}^{(1)}\right\rangle}{\Delta_{21}^2}+$$
$$+D_{21}\frac{\left(\left\langle u^{(2`)},\xi^{(21)}\right\rangle-c^2\tau_{21}\right)\dot{u}_\alpha^{(1)}-\left\langle u^{(2)},\dot{u}^{(1)}\right\rangle\xi_\alpha^{21}}{\left[c^2\tau_{21}-\left\langle u^{(1`)},\xi^{(21)}\right\rangle\right]^2}+$$
$$+\frac{D_{21}}{\Delta_{21}^2}\cdot\frac{\left(\left\langle u^{(2`)},\xi^{(21)}\right\rangle-c^2\tau_{21}\right)u_\alpha^{(1)}\left\langle u^{(1)},\dot{u}^{(1)}\right\rangle}{\left[c^2\tau_{21}-\left\langle u^{(1`)},\xi^{(21)}\right\rangle\right]^2}+ \qquad (3_{2\alpha})$$
$$\left.+\frac{D_{21}}{\Delta_{21}^2}\cdot\frac{\left(c^2-\left\langle u^{(2`)},u^{(1)}\right\rangle\right)\xi_\alpha^{(21)}\left\langle u^{(1)},\dot{u}^{(1)}\right\rangle}{\left[c^2\tau_{21}-\left\langle u^{(1`)},\xi^{(21)}\right\rangle\right]^2}\right\}.$$

Recall that in the above equations $(3_{1\alpha})$

$$u^{(1)}=u^{(1)}(t),\ u^{(2)}=u^{(2)}(t-\tau_{12})$$

while in $(3_{2\alpha})$

$$u^{(2)}=u^{(2)}(t),\ u^{(1)}=u^{(1)}(t-\tau_{21})\ .$$

We note that the delay functions $\tau_{pq}(t)$ satisfy functional equations (*) for $t\in(-\infty,\infty)$. For $t\le 0$ $u_\alpha^{(p)}(t)$ are prescribed functions: $u_\alpha^{(p)}(t)=\bar{u}_\alpha^{(p)}(t),\ t\le 0$, where $\bar{u}_\alpha^{(p)}(t)=\frac{d\bar{x}_\alpha^{(p)}(t)}{dt},\ t\le 0$.

This means that for prescribed trajectories $(\bar{x}_1^{(1)}(t),\bar{x}_2^{(1)}(t),\bar{x}_3^{(1)}(t)),(\bar{x}_1^{(2)}(t),\bar{x}_2^{(2)}(t),\bar{x}_3^{(2)}(t))$ for $t\le 0$ one has to find trajectories, satisfying the above system of equations for $t>0$.

(We recal $x_\alpha^{(p)}(t)=x_{\alpha 0}^{(p)}+\int_0^t u_\alpha^{(p)}(s)ds$, where $x_{\alpha 0}^{(p)}$ are the coordinates of the initial positions.)

Kepler problem in polar coordinates

In what follows we consider plane motion in Ox_2x_3 coordinate plane for above equations. We suppose that the first particle P_1 is fixed at the origin $O(0,0,0)$, that is,

$$P_1:\left|\begin{array}{l}x_1^{(1)}(t)=0\\ x_2^{(1)}(t)=0,\quad t\in(-\infty,\infty)\ .\\ x_3^{(1)}(t)=0\end{array}\right.$$

It follows by necessity $\left|\begin{array}{l}\bar{x}_1^{(1)}(t)=0\\ \bar{x}_2^{(1)}(t)=0\ .\\ \bar{x}_3^{(1)}(t)=0\end{array}\right.$

Passing to the polar coordinates we can put

$$P_1:\left|\begin{array}{l}x_1^{(2)}(t)=0\\ x_2^{(2)}(t)=\rho(t)\cos\varphi(t)\\ x_3^{(2)}(t)=\rho(t)\sin\varphi(t)\end{array}\right.\quad\text{, where } \rho(t)>0\ .$$

After transformations made in (Angelov, 2000) we obtain the following second order system:

$$\ddot{\rho}(t)=\rho(t)\dot{\varphi}^2(t)+\frac{Q_2}{c^3}\cdot\frac{\left[c^2-\dot{\rho}^2(t)\right]\sqrt{c^2-\dot{\rho}^2(t)-\rho^2(t)\dot{\varphi}^2(t)}}{\rho^2(t)}$$
$$\ddot{\varphi}(t)=-\frac{2\dot{\rho}(t)\dot{\varphi}(t)}{\rho(t)}\cdot\left[1+\frac{Q_2\sqrt{c^2-\dot{\rho}^2(t)-\rho^2(t)\dot{\varphi}^2(t)}}{2c^3\rho(t)}\right] \qquad (4)$$

for $t>0$ and initial conditions

$$\rho(0)=\rho_0,\ \dot{\rho}(0)=\dot{\rho}_0,\ \varphi(0)=\varphi_0,\ \dot{\varphi}(0)=\dot{\varphi}_0\ .$$

On the other hand beginning with the original form of Synge equations (Angelov, 2002) we obtain for Kepler problem the following equations of motion:

$$\frac{d(\gamma_2 u_\alpha^{(2)})}{dt}=\frac{Q_2\xi_\alpha^{(21)}}{\rho^3}\quad(\alpha=1,2,3)\ . \qquad (5_\alpha)$$

But $\xi^{(21)}=(0,\rho(t)\cos\varphi(t),\rho(t)\sin\varphi(t))$. Then integrating (5_α) from 0 to *t* we have

$$\gamma_2(t)u_\alpha^{(2)}(t)-\gamma_2^0 u_\alpha^{(2)}(0)=Q_2\int_0^t\frac{\xi_\alpha^{(21)}(s)}{\rho^3(s)}ds \text{ or}$$

$$\left|\begin{array}{l}\dot{\rho}(t)\cos\varphi(t)-\rho(t)\dot{\varphi}(t)\sin\varphi(t)=\frac{1}{\gamma_2(t)}\left[Q_2\int_0^t\frac{\cos\varphi(s)}{\rho^2(s)}ds+C_2\right]\\ \dot{\rho}(t)\sin\varphi(t)+\rho(t)\dot{\varphi}(t)\cos\varphi(t)=\frac{1}{\gamma_2(t)}\left[Q_2\int_0^t\frac{\sin\varphi(s)}{\rho^2(s)}ds+C_3\right]\end{array}\right. \qquad (6)$$

where $C_2=\gamma_2^0\left(\dot{\rho}_0\cos\varphi_0-\rho_0\dot{\varphi}_0\sin\varphi_0\right)$,

$C_3=\gamma_2^0\left(\dot{\rho}_0\sin\varphi_0+\rho_0\dot{\varphi}_0\cos\varphi_0\right)$,

$$\gamma_2^0=\frac{1}{\sqrt{1-\frac{1}{c^2}(\dot{\rho}_0^2+\rho_0^2\dot{\varphi}_0^2)}}\ .$$

The systems (6) and (5_α) are equivalent.

Indeed, the right-hand side of (5_α) is the vector-function $\vec{F}(t)=\left(\frac{\cos\varphi(t)}{\rho^2(t)},\frac{\sin\varphi(t)}{\rho^2(t)}\right)$ and $\left|\vec{F}(t)-\vec{F}(t_0)\right|^2=$ $=\frac{1}{\rho^4(t)}+\frac{1}{\rho^4(t_0)}-\frac{2\cos(\varphi(t)-\varphi(t_0))}{\rho^2(t)\rho^2(t_0)}\underset{t\to t_0}{\to}0$, that is, $\vec{F}(t)$ is continuous vector-function – the necessary and sufficiently condition for equivalence of (6) and (5_α).

On the other hand $\dot{\gamma}_2=\frac{d}{dt}\left(\frac{c}{\Delta_2}\right)=\frac{c}{\Delta_2^3}\langle u,\dot{u}\rangle$,

$$\frac{d}{dt}\left(\gamma_2 u_\alpha^{(2)}\right)=\dot{\gamma}_2 u_\alpha^{(2)}+\gamma_2\dot{u}_\alpha^{(2)},\ (\alpha=2,3)$$

Then from (5_α) we obtain the system (with $u\equiv u^{(2)}$):

$$\left|\begin{aligned}&\frac{c}{\Delta_2^3}\langle u,\dot{u}\rangle u_2+\frac{c}{\Delta_2}\dot{u}_2=\frac{Q_2\cos\varphi}{\rho^2}\\&\frac{c}{\Delta_2^3}\langle u,\dot{u}\rangle u_3+\frac{c}{\Delta_2}\dot{u}_3=\frac{Q_2\sin\varphi}{\rho^2}.\end{aligned}\right.\qquad(7)$$

Multiplying the first equations of (7) by u_2, the second - by u_3 and summing we obtain:

$\frac{c}{\Delta_2^3}\langle u,\dot{u}\rangle\left(\langle u,u\rangle+\Delta_2^2\right)=\frac{Q_2}{\rho^2}\quad(u_2\cos\varphi+u_3\sin\varphi)$, that is $\frac{c}{\Delta_2^3}\langle u,\dot{u}\rangle=\frac{Q_2\dot{\rho}}{c^2\rho^2}$.

Thus we have $\dot{\gamma}_2=\frac{Q_2\dot{\rho}}{c^2\rho^2}$.

Multiplying the first equations of (7) by $\cos\varphi$, the second – by $\sin\varphi$ and summing, and next multiplying the first equations of (7) by $-\sin\varphi$, the second - by $\cos\varphi$ and summing, we obtain the system $\left|\begin{aligned}&\dot{\gamma}_2\dot{\rho}+\gamma_2(\ddot{\rho}-\rho\dot{\varphi}^2)=\frac{Q_2}{\rho^2}\\&\dot{\gamma}_2\rho\dot{\varphi}+\gamma_2(2\dot{\rho}\dot{\varphi}+\rho\ddot{\varphi})=0\end{aligned}\right.$, or

$(\dot{\gamma}_2=\frac{Q_2\dot{\rho}}{c^2\rho^2})$

$$\left|\begin{aligned}&\gamma_2(\ddot{\rho}-\rho\dot{\varphi}^2)=\frac{Q_2}{c^2\rho^2}(c^2-\dot{\rho}^2)\\&\gamma_2(\rho\ddot{\varphi}+2\dot{\rho}\dot{\varphi})=-\frac{Q_2}{c^2\rho}\dot{\rho}\dot{\varphi}\end{aligned}\right.\qquad(8)$$

The final system (8) is equivalent to the system (4), since $\gamma_2=\frac{c}{\Delta_2}=\frac{c}{\sqrt{c^2-\dot{\rho}^2-\rho^2\dot{\varphi}^2}}$.

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