ARCHIMEDES AND ARCHIMEDES‘ CONSTANT

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ABSTRACT. Archimedes' constant, the number Pi (π) has fascinated mathematicians since ancient times and has since been the subject of their research. This paper shows a short history of the number Pi (π) to the appearance of the Greek mathematician Archimedes. This paper also considers Archimedes' contribution with regard to the definition of Archimedes' constant, the number Pi (π).

Keywords: Mathematics, Archimedes constant, number Pi, Archimedes.

АРХИМЕД И АРХИМЕДОВАТА КОНСТАНТА

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РЕЗЮМЕ. Константата на Архимед, числото π очарова математиците от древни времена и оттогава е обект на техните изследвания. Докладът представя накратко историята на числото π и появата на гръцкия математик Архимед. Разгледан е също приносът на Архимед при определяне на Архимедовата константа, числото π.

Ключови думи: Математика, Архимедова константа, числото π, Архимед.

Introduction

"The number Pi has been known since ancient times and it shows the ratio of the circumference and the diameter of a circle, and is usually written down as 3,14. The role of this number was specially highlighted 1706, in British mathematician William Jones' (1675-1749) work called Synopsis Palmariorum Mathseos. The number Pi was probably inspired by the Greek work for circumference, περιφερια (periphery). This designation was gradually popularized in the works of mathematician Leonhard Euler. In literature on the History of mathematics, Pi is also known as Archimedes' constant, but also as Ludolfs' number. In 1761, Johann Heinrich Lambert (1728-1777) was the first to prove that Pi is an irrational value. Adrien-Marie Legendre (1752-1833) also produced a proof in 1794, that Pi raised to the square is an irrational constant. But it was only in 1882 that Ferdinand von Lindemann (1852-1939) managed to prove that Archimedes' constant also is a transcendental number, which means that it cannot be the root of integer polynomials. This result showed that one of the best known mathematical problems in history, i.e. squaring the circle, cannot be solved rationally."[1]

The history of the number Pi before Archimedes

The first known calculations of the number Pioriginate from ancient Egypt, approx. 2500 BC. These calculations have not been documented anywhere, but it is believed that they were used to build the Great Pyramid in Gizeh, because the pyramid has a base that measures 1760 cubits and a height of 280 cubits, and if we calculate the ratio of these two values, we obtain the following:

$$\frac{1760}{280} \approx 6.285714285714286 \approx 2\pi,$$

$$\pi \approx 3.142$$

And this calculation is one of the most precise in ancient History, the error is only 0.04%.

The first known record of number Pi was made by Egyptian writer Ahmosis, approx. 1750 BC in the Rhind papyrus. In that record it has the value 3.16. This value is much more inaccurate than the value originating from Egypt, with an error of 0.6%. In the 19th century BC, Babylonian mathematician used Pi with a value of approx. 25/8, where the error was 0.5%. In the ninth century BC, Indian mathematician Yajnavalkya calculates the value of Pi as 3.13888… with an error of only 0.09%. Later, Chinese mathematician Liu Hui, in the third century BC, gives a very accurate value of Pi, using a method similar to that of Archimedes. He determines Pi

[1]Letić Dušan, Cakić Nenad, Davidović Branko, Matematičke konstante, Beograd, 2010.

between 3.141024 and 3.142708, which on average is 3.141864, with an error of less than 0.01%. Later, he gives an even more accurate approximation: $\pi \approx \frac{3927}{1250} = 3.1416$.

Archimedes life (287 BC-212 BC)

Archimedes (*Fig. 1*) was a Greek mathematician, physicist, engineer, inventor and astronomer from Syracuse in Sicily. The year of his birth is based on the claims of historian John Tzetzeshat Archimedes lived 75 years. Archimedes' biography was written by his friend Heracleides, but this work has been lost, leaving the details of his life unclear. For example, it is not known whether he was ever married or had children. Archimedes' father was mathematician and astronomer Phidias (not the sculptor Phidias). When Archimedes was born, Phidias was a relatively poor citizen, one of many in Syracuse. But he was not poor for very long, as their relative Hierobecameruler of the city. Phidias taught his son everything he knew. Form early childhood, he developed in his the love for mathematics, mechanics and astronomy.

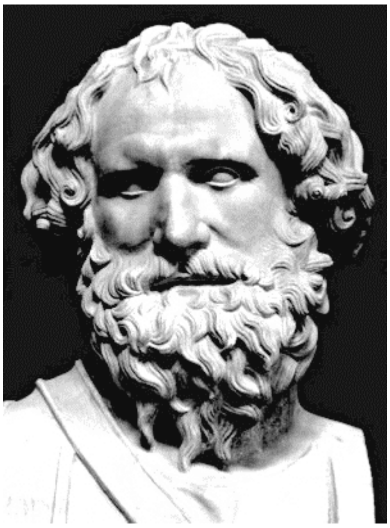

Fig. 1. Archimedes from Syracuse (287-212 BC)

Archimedes' father Phidiashad the following idea: he would give his son all of his knowledge, who should later do with it as he pleases. Archimedes quickly adopted his father's knowledge, which for him was only the beginning of his learning. His spirit was looking for more knowledge and learning, which he could not find in Syracuse. Therefore he went to Alexandria, which was ruled by the Ptolemeian dynasty, who had founded the famous Library of Alexandria. At that time, Alexandria was the centre of natural sciences, which then included astronomy, mathematics, medicine and philology. In Alexandria, Archimedes did not achieve what he probably could, - he did not a member of the court, who through his works will praise the ruling dynasty. He was only interested in mathematics.

Many talented mathematicians worked at the Library of Alexandria. The most versatile of them was the brilliant Eratosthenes, a future friend of Archimedes. An unwritten rule sad that any discovery, prior to publication, has to be sent to another mathematician to be checked. And so, Archimedes π and Eratosthenes ex changed numerous letters until Archimedes' death, and these letters contained almost all discoveries of the two. After coming back to Syracuse, Archimedes was into astronomy in the beginning.

During the siege of Syracuse, Archimedes invented numerous weapons which were used against the Roman fleet that attacked Syracuse in 215 BC. Neither Archimedes' genius, nor the defense of the citizens of Syracuse did prevent the Romans from Syracuse in 212 BC, when Archimedes died.

"It is also believed that Archimedes designed a huge burning mirror in the shape of a paraboloid, which was used to put enemy vessels fire. With a little knowledge in physics and mathematics it is possible to calculate that the length of the *latus rectum* (a straight line going through the focal point normally to the axis) is equal to parameter p in the parabolic equation $y^2 = px$. As the focal point of the parabola $y^2 = px$ is in the point (p/4,0), and assumed that an enemy ship is at a distance of 50 meters from the city wall sand that it is placed in the focal point of Archimedes' mirror, at p/4=50 meters, brings us to the conclusion that its diameter must have been p=200 meters. It is clear that in those times (neither is it today) it was not possible to build a mirror of this size."[2]

Archimedes' death

Archimedes died in 212 BC during the second Punic war, when Roman forces under general Marcus Claudius Marcellus took Syracuse after two years of siege (Fig. 2). It is told that his last words were "*Noli turbare circulos meos!"* (Do not touch my circles!) (Greek: μη μου τους κυκλους ταραττε).

Archimedes' death during the siege of Syracuse, is known to us because of Plutarch's biography of general Marcellus. Plutarch tells the following*: "Marcellus was mostly shaken by the unfortunate death of Archimedes. He was studying a geometric shape alone. And, focused on his thoughts and with his eyes fixed to the subject of his study, he neither did notice the Roman breakthrough, nor the fall of the city, when a soldier suddenly approached him and ordered Archimedes to follow him to meet Marcellus, but Archimedes would not to go until he solved the problem and proved the solution. Others tell that the Roman approached him with the intention to kill him immediately with his sword but that Archimedes, seeing him, kindly asked him to wait a little, so that he result would not be unfinished and incomplete, but the soldier did not listen and killed him. There is also a third version is that Archimedes was killed by soldiers who thought that he was carrying gold, when he was taking his mathematical devices, sun dials, balls and angle meters which makes it possible to measure the size of the sun, to show them to Marcellus. However, all historians agree that Marcellus was very sad and that he turned his head away from his killer, and instead rewarded Archimedes' relatives."*[3]

[2]Petković Miodrag, Petković Ljiljana, Matematički vremeplov –prilozi za istoriju matematike,Zmaj, Novi Sad,2006. (str.2.).
[3]Plutrarh, Uporednizi vo topisi knjiga II (Zivot Marcelov),(strana 443), August Cesarac, Zagreb,1988

Fig. 2. Archimedes' death –painted by Edouard Vimont (1846-1930)

Fig. 3. Remains of Archimedes' tomb in Syracuse

It is not rare that appropriate words or images are engraved on gravestones of famous people. These engravings usually describe the greatest achievements, life style or part of the characters of these people. An example is Archimedes' monument in Sicily. Roman statesman and orator Cicero, served as a quaestor in Sicily, 75 BC, 137 years after Archimedes' death. He heard the stories about Archimedes' death and wanted to visit his grave, but none of the locals was able to show the location of Archimedes' tomb. At last he found the tomb near the Agrigata gate in Syracuse, in a ruined state and recognized it by the "*small pillar which was showing behind a bus hand on which ball and a cylinder were shown*"[4]. Cicero restored the tomb (*Fig. 3*).

Archimedes contribution to the development of science

Archimedes was one of the leading scientists of Antiquity. Besides Newton and Gauss, he is considered one of the greatest mathematicians. Archimedes has also significant contributions in mechanics and astronomy. He was the first to clearly understand the term of specific weight and in his work "About floating bodies" established the principles of hydrostatics. He gained fame because of great achievements in the field of mechanics, which helped Syracuse to withstand the Roman siege. Archimedes was also an astronomer, thus he found out that one year has 365 ¼ days.

His works „On the equilibrium of planes" (two books), "Squaring of the parabola", "On the sphere and the cylinder" (two Books), "On spirals", "On Conoids and Spheroids", "On floating bodies" (two Books), "On the Measurement of the Circle", "The Sand Reckoner" and "The Method of mechanical theorems".

By discovering the following principle:when a body entirely or partially immersed in a fluid, it is subject to a force acting up, which has the effect of a temporary weight loss, equal to the weight of the deplaced fluid Archimedes laid the foundations for hydrostatics.

Among his practical inventions from the field of hydraulics, the Archimedes screw should be mentioned, which makes possible to transport water from a lower to a higher-level; this device was used for irrigation. (Fig. 4).

Fig. 4. Archimedes-painted 1629 by Domenico Fetti

Archimedes developed the well-known stereometry theorem, which can be obtained from the equilibrium property and incorporated it in his famous work "On the sphere and the cylinder", which states:
The ratio of the volume of the rectangular circular cylinder, the height of which is equal to its diameter and the diameter of the sphere inscribed in that cylinder, is 3:2, which also is equal to the ratio of its surface areas.

This was one of Archimedes' favorite theorems, and following his request, it was engraved on his tombstone. Besides these words, a drawing with three geometrical bodies was also engraved. The figure shows a cone and a sphere inscribed in the cylinder, the height of which is equal to the diameter of the base. It is known that the ratio of the volume of the cone, sphere and cylinder of these dimensions is 1:2:3. This figure symbolizes Acrhimedes' work in the field of geometry, which he considered his greatest success (*Fig. 5*). Archimedes discovered the correct formulae for calculating the surfaces of many flat shapes and volumes of geometrical bodies.

[4]Lučić Zoran, Ogledi iz istorije Antičke geometrije, Službeni glasnik, 2009 (str.281)

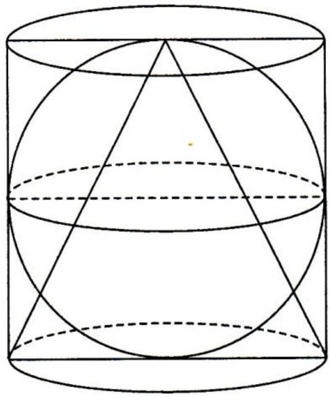

Fig. 5. Archimedes' cone and sphere inscribed in a cylinder, the height of which is equal to the diameter of the base

Archimedes' contribution to the definition of π

Archimedes' calculation of the ratio between the circumference and the diameter of a circle and placing the value of that ratio in a range between 22/7 and 223/71 significantly contributed to the development of geometry. This ratio was explained and calculated in the treatise "*On the measuring of circle*". However, he did not name this ratio, nordid he give it a different symbol. After his death, mathematicians named this number Archimedes' constant and started calling it Ludolfs' number in the 17thcentury, after mathematician Ludolf van Ceulen, who calculated thirty-five decimals of this ratio. 1706,English mathematician William Jones designated this ratio, or Archimedes' constant, or Ludolfs' number, with the Greek letter "π", since when it has been called the number Pi.

Discovering the circumference of a polygon with 2n sides, is not Archimedes' direct discovery: this circumference was calculated by Antiphontes. But Archimedes simplified and rationalized the figure and, seemingly, obtained a result through simpler and more accurate calculation. Instead of drawing many circumscribed polygons, he aligns the half-sides of polygons with one and the same key (Fig. 6). Indeed, if AB is half of a side of a circumscribed n-polygon; one should only split in half the angle BOA and draw a bisector OC, to obtain the segment AC, equal to a half-side of a circumscribed 2n polygon. Bz then dividing angle COA in half and drawing bisector OD, we obtain segment AD which is equal to the half-side of the circumscribed 4n polygon, and so on.

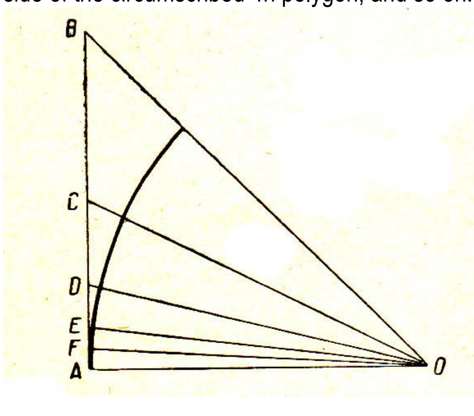

Fig. 6. Alignment of half-sides of circumscribed polygons with one and the same key

If AB is the half-side of a circumscribed hexagon, and accordingly angle BOA is equal to a third of aright angle, then

$$OA:AB=\sqrt{3}:1>265:153 \quad (1)$$

$$OB:AB=2:1=306:153 \quad (2)$$

How Archimedes had the idea to use the approximate value 265:153 for $\sqrt{3}$, we do not know: it was either already calculated by his predecessors, or he made this calculation himself in one of his lost works on Arithmetic; Archimedes just gives us the result without any explanation.

To determine the length of the half-side AC of a 12-sided polygon, or AD of a 24-sided AD, etc., Archimedes uses the following theorem: the bisector divides the base line into parts that are proportional to the sides:

OB:OA = BC:CA
(OB+OA):OA = (BC+CA):CA
(OB+OA):OA = BA:CA
(OB+OA):BA = OA:CA

But adding (1) to (2), we obtain

(OB+OA):BA=(265+306):153=571:153

Therefore, we have

$$OA:CA=571:153 \quad (3)$$

or
OA=571 part, CA=153 parts.

To determine the OC:CA ratio, take into consideration that OC is the hypotenuse of triangle OAC, the legs of which are CA and OA, and therefore we have

$OC^2=CA^2+OA^2$

But from (3) we have
$OA^2+CA^2=(571^2+153^2)$ parts.
$OC^2=(571^2+153^2)$ parts
$OC^2:CA^2=(571^2+153^2):153^2=349450:23409$,

Base o that Archimedes writes
$OC:CA>591\frac{1}{3}:153$

How Archimedes found the square root of 349450, is not known.

Continuing to apply the same method in the further side-doubling procedure, he obtains the following value for the half-side of a 24-sided polygon

$OA:DA > 1162\frac{1}{8} : 153$,

$OD:DA > 1172\frac{1}{8} : 153$

Where $1172\frac{1}{8}$ is the square root of $1373943\frac{33}{64}$; In this case, Archimedes does not explain either how this square root was calculated.

Continuing with the half-side AE of a 48-sided polygon, Archimedes obtains the following

$OA:EA > 2334\frac{1}{4}:153$

$OE:EA > 2339\frac{1}{4}:153,$

And for the half-side OF of a 96-sided polygon:

$OA:AF > 4673\frac{1}{2}:153.$

But the ratio of radius OA and the half-side of a 96-sided polygon is equal to the ratio between the diameter and whole side of a 96-sided polygon; which means that the ratio of the diameter and the whole circumference is

$> 4673\frac{1}{2} : (153x96) > 4673\frac{1}{2} : 14688$

Therefore, the ratio between the circumference of the 96-sided polygon and the diameter is

$$< 14688: 4673\frac{1}{2} < 3 + \frac{667\frac{1}{2}}{4673\frac{1}{2}} < 3\frac{1}{7}.$$

It is easy to see in the figure that Archimedes here establishes a rule for gradually calculating

$\mathrm{ctg}\,\alpha, \mathrm{ctg}\,\frac{\alpha}{2}, \mathrm{ctg}\,\frac{\alpha}{4}, \ldots$

To determine the lower boundary (Fig. 7) Archimedes begins with the side of the inscribed n-sided polygon AB. If point B is connected with the opposite end $A_1$of diameter AA_1we obtain a right angled triangle. If we split angle AOB in half and draw a bisector OC, then side AC will obviously be the side of a 2n-sided polygon. However, by drawing CA_1, we can see that in this case, angle AA_1B will be divided in half (based on $\angle AA_1B = \frac{1}{2}\angle AOB$, a $\frac{1}{2}\angle AA_1C = \frac{1}{2}\angle AOC$); Therefore, even when we split the angle in half at A_1,we always get a side of a polygon with a doubled number of sides.

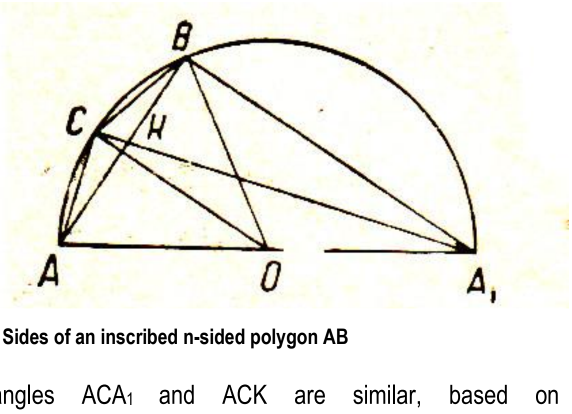

Fig. 7. Sides of an inscribed n-sided polygon AB

Triangles ACA_1 and ACK are similar, based on $\angle CAK = \angle BA_1C$, because they lie on the same arc CB, a $\angle BAC = \angle CA_1A$ and therefore it is $\angle CAK = \angle CA_1A$; where $\angle ACA_1$ as the common angle. Therefore, we have

$$CA_1:AC = AC:CK = AA_1:AK \quad (4)$$

But A_1C is the bisector $\angle BA_1A$; and so we have (per mutando).

$$AA_1:AK = A_1B:BK; \quad (5)$$

From (1) and (2) utomnesadomnes, itaunus ad unum:

$CA_1:AC = (AA_1+A_1B):(AK+BK),$
$CA_1:AC = (AA_1+A_1B):AB,$

Where, in the case of a hexagon
$CA_1:AC=(2+\sqrt{3}):1$
etc.

In the same way as in the case with the circumscribed 96-sided polygon, for the inscribed 96-sided polygon we obtain the following ratio of circumference and diameter

$> (66x96):2017\frac{1}{4} > 6336: 2017\frac{1}{4} > 3\frac{10}{71}.$

And here it is also necessary to take the square root of very large numbers, but Archimedes gave no explanation how he did that.

It is easy to see that the values he gradually obtained, are

$\mathrm{cosec}\,\alpha, \mathrm{cosec}\,\frac{\alpha}{2}, \mathrm{cosec}\,\frac{\alpha}{4}, \ldots$

The deviation margin is obviously the same[5]

$3\frac{1}{7} - 3\frac{10}{71} = \frac{1}{497} \approx 0.002.$

Instead of a conclusion

The fame of Archimedes' work was maintained by the Arabs: Ishak Ibn Hunan, the translator of Archimedes' masterpiece: «On the sphere and the cylinder», Tabit Ibn Kurah, the translator of the treatise: «Measurement of the circle», Almohtasoabil Hasan, al-Jalil as Sijzi, al-Kuhi, al Mahani, al-Biruni, and especially Omar Hajjam, the famous poet Rubaije, and the greatest Arab mathematician Muhamed Ben-Musa al-Khwarizmi[6].

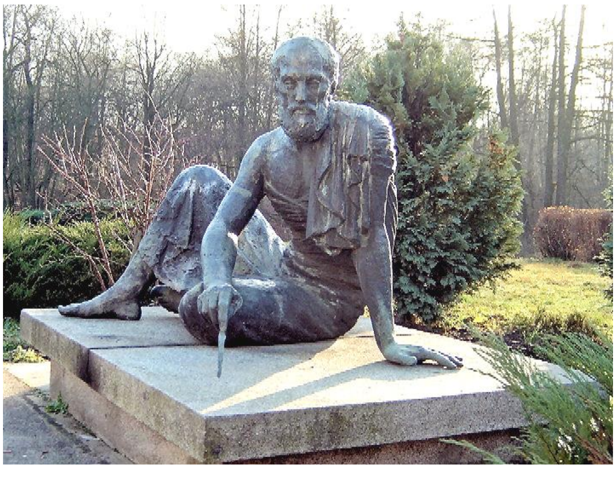

Fig. 8. The monument of Archimedes, built in 1972, in front of Berlin observatory (by Gerhard Thieme)

[5]Lurje S. J. , Arhimed, Prosveta, Beograd 1952 (str. 172, 173, 174,175,176)
[6]Risojević Ranko, Veliki matematičari, Nolit, 1981. (str. 22.)

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