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# MINIMAL COMMUTATIVITY OF COMPOSITIONS OF OPERATORS OF MIXED TYPE - I

***Miryana Hristova***

*University of National and World Economy, Department of Mathematics, Studentski grad, 1700 София, Bulgaria*
*E-mail: miryana_hristova@abv.bg*

**ABSTRACT.** The operators $Hf(z) = H_{p,q}f(z) = \frac{d^p}{dz^p}\left(z^q\int_0^z f(\zeta)d\zeta\right)$ are considered with non-negative integer parameters $p, q \in \boldsymbol{Z}_+$ in the case when $p \le q+1$ in the space $A_0$ of the functions analytic in neighbourhoods of the origin $z = 0$ of the complex plane $\boldsymbol{C}$. Using the power series descriptions of the commutants of compositions of operators of the type $H_{p,q}$ with different parameters $p$ and $q$ from previous author's papers, here the question about the minimal commutativity (in the sense of (Raichinov 1979)) of compositions is considered.

**МИНИМАЛНА КОМУТАТИВНОСТ НА КОМПОЗИЦИИ ОТ ОПЕРАТОРИ ОТ СМЕСЕН ТИП**

***Миряна Христова***

*Университет за национално и световно стопанство, Катедра Математика, Студентски град, 1700 София, България*

**РЕЗЮМЕ.** Операторите $Hf(z) = H_{p,q}f(z) = \frac{d^p}{dz^p}\left(z^q\int_0^z f(\zeta)d\zeta\right)$ са разгледани с неотрицателни цели параметри $p, q \in \boldsymbol{Z}_+$ в случая, когато $p \le q+1$, в пространството $A_0$ на функциите аналитични в околности на координатното начало $z = 0$ на комплексната равнина $\boldsymbol{C}$. Използвайки описанието чрез степенни редове на комутантите на композиции на оператори от вида $H_{p,q}$ с различни параметри $p$ и $q$ от предишни свои статии, авторът разглежда тук въпроса за минималната комутативност (в смисъл на (Райчинов 1979)).

## Introduction

Let $A_0$ be the space of functions analytic in (possibly different) neighbourhoods of the origin $z = 0$ in the complex plane $\boldsymbol{C}$ or its subspace $S$ of the polynomials of the complex variable $z$. We want to consider a generalization of the usual operator of integration $\int_0^z f(\zeta)d\zeta$ multiplying it by a non-negative power $z^q$ and then differentiating $p$ times, i.e. we consider the operators of mixed type

$$H_{p,q}f(z) = \frac{d^p}{dz^p}\left(z^q\int_0^z f(\zeta)d\zeta\right), \quad p, q \in \boldsymbol{Z}_+. \tag{1}$$

In (Hristova 2012) the commutational properties of a single operator $H_{p,q}$ in the case $p < q+1$ were investigated, and in (Hristova 2013a, 2013b) the case $p = q+1$ is presented. Here we will combine the results from these papers to describe the commutants of compositions of operators of the type $H_{p,q}$ with different parameters $p$ and $q$ when they increase or preserve the powers. We will discuss also the question about the minimal commutativity of compositions (in the sense of (Raichinov 1979)).

Let us represent first the action of only one operator $H_{p,q}$ on a single power $z^k$:

$$H_{p,q}z^k = \\ = \frac{1}{k+1}(k+q+1)((k+q+1)-1)\dots((k+q+1)-p+1)z^{(k+q+1)-p}. \tag{2}$$

Denoting $q-p+1$ by $\alpha$, we have $\alpha \ge 0$ and can write shortly

$$H_{p,q}z^k = a_k z^{k+\alpha}; \quad a_k \overset{\text{def}}{=} \frac{1}{k+1}\cdot\frac{(k+q+1)!}{(k+\alpha)!} \ne 0, \\ \alpha \overset{\text{def}}{=} q-p+1 \ge 0. \tag{3}$$

Now an arbitrary power $H_{p,q}^r$ of $H_{p,q}$ acts on $z^k$ as

$$H_{p,q}^r z^k = a_k a_{k+\alpha} \dots a_{k+(r-1)\alpha} z^{k+r\alpha}, \quad a_l = \frac{1}{l+1} \cdot \frac{(l+q+1)!}{(l+\alpha)!} \neq 0. \quad (4)$$

In order to avoid writing the long products in (4) we will use again a short representation denoting them by one letter:

$$\beta_k \overset{\text{def}}{=} a_k a_{k+\alpha} \dots a_{k+(r-1)\alpha}, \quad a_l = \frac{1}{l+1} \cdot \frac{(l+q+1)!}{(l+\alpha)!} \neq 0. \quad (5)$$

and then we can write simply

$$H_{p,q}^r z^k = \beta_k z^{k+r\alpha}, \quad \beta_k = \prod_{t=0}^{r-1} a_{k+t\alpha}, \qquad a_l = \frac{1}{l+1} \cdot \frac{(l+q+1)!}{(l+\alpha)!} \neq 0. \quad (6)$$

In fact, if $y(z) = \sum_{k=0}^{\infty} c_k z^k$ is an analytic function from $A_0$ with coefficients $c_k \overset{\text{def}}{=} \frac{y^{(k)}(0)}{k!}$, then we have the short representation

$$H_{p,q}^r y(z) = \sum_{k=0}^{\infty} c_k \beta_k z^{k+r\alpha} \quad (7)$$

with $\beta_k$ from (5) and (6) and $\alpha$ from (3).

Let us give some definitions:

**Definition 1.** It is said that a continuous linear operator $L$ commutes with a fixed operator $M$, if $LM = ML$. The set of all such operators is called the *commutant* of $M$ and will be denoted by $C_M$.

**Definition 2.** It is said that a continuous linear operator $T$ is *generated* by an operator $M$, if $T$ is a polynomial of $M$ with complex coefficients, i.e. $T = \sum_{n=0}^{\infty} d_n M^n$, $d_n \in \boldsymbol{C}$. The set of all operators generated by $M$ will be denoted by $G_M$.

Obviously every operator $T$, which is generated by $M$, i.e. $T \in G_M$, also commutes with $M$, i.e. $T \in C_M$, and hence $G_M \subset C_M$. The opposite inclusion $G_M \supset C_M$ is, in general, not true. Therefore the following definition is natural:

**Definition 3.** (Raichinov 1979) An operator $M$ is called minimally commutative if $G_M \supset C_M$, i.e. if the commutant $C_M$ consists only of operators $T$ generated by $M$ and hence if $C_M = G_M$.

In general we can consider compositions

$$H^r = H_1^{r_1} H_2^{r_2} \dots H_s^{r_s}, \quad (8)$$

where $H_1 = H_{p_1,q_1}, H_2 = H_{p_2,q_2}, \dots, H_s = H_{p_s,q_s}$ are operators of the form (1), $r_1, r_2, \dots, r_s$ are nonnegative integers, and $r = (r_1, r_2, \dots, r_s)$ is considered as a multipower.

In the papers (Hristova 2013c -2013f) the author considers for the sake of simplicity only compositions of two operators

$$H^r = H_1^{r_1} H_2^{r_2}, \qquad r = (r_1, r_2). \quad (9)$$

The description of the commutants of compositions is given there in different cases: when the powers are preserved or increased by both operators and also in the mixed cases when one of the operators increases, while the other one preserves the powers. It is convenient to define the numbers

$$\alpha_j \overset{\text{def}}{=} q_j - p_j + 1 \geq 0, \qquad j = 1, 2, \quad (10)$$

which show how each of the operators in the composition changes the powers of the complex variable $z \in \boldsymbol{C}$.

In the general case of composition of more than two operators the reasonings are the same but the written form of the results becomes more complicated.

Let us note that descriptions of commutants are made by many mathematicians. In the references of this paper we have included only a very small part of the publications related to the commutants of operators similar to the one considered here, see all refferences. Additional huge number of publications related to commutants can be found in the bibliographies of the cited monographs.

## The case of preserving the powers

### Description of the commutant

The description of the commutant in the case of composition of operators preserving the powers is proved in another paper (Hristova 2013c). The interesting fact is that it remains the same as the one for a single operator given in paper (Hristova 2013a).

**Theorem 1.**

Let the operators $H_1$ and $H_2$ be of the type (1) with $\alpha_j = q_j - p_j + 1 = 0$, $j = 1, 2$. We can fix $q_1 \neq q_2$ and then to express $p_j = q_j + 1$, $j = 1, 2$, writing

$$H_j y(z) = H_{q_j} y(z) = H_{q_j+1,q} y(z) = \\ = \frac{d^{q_j+1}}{dz^{q_j+1}} \left( z_j^q \int_0^z y(\zeta) d\zeta \right), \; j = 1, 2. \quad (11)$$

Then a linear operator $L: A_0 \to A_0$ commutes with the composition operator $H^r = H_1^{r_1} H_2^{r_2}$, $r_j \geq 1$, $j = 1,2$, if and only if it has the form

$$Ly(z) = \sum_{k=0}^{\infty} \frac{y^{(k)}(0)}{k!} d_k z^k, \qquad (12)$$

where $\{d_k\}_{k=0}^{\infty}$ is an arbitrary sequence of complex numbers, but such that the series in (12) converges.

For the sake of completeness we will give here only a

**Sketch of the proof**

A short expression of the action of either of the two operators $H_j$, $j = 1,2$, on an arbitrarily fixed single power $z^k$ of the complex variable $z$ is

$$H_j z^k = \beta_{j,k} z^k = (k + q_j + 1)(k + q_j) \ldots (k+2)\, z^k. \qquad (13)$$

Then the action of the composition $H^r = H_1^{r_1} H_2^{r_2}$ on $z^k$ can be written as

$$\begin{aligned} H^r z^k &= (H_1^{r_1} H_2^{r_2}) z^k = H_1^{r_1} (H_2^{r_2} z^k) = \\ H_1^{r_1} &(\beta_{2,k}^{r_2} z^k) = \beta_{1,k}^{r_1} \beta_{2,k}^{r_2} z^k. \end{aligned} \qquad (14)$$

If $L: A_0 \to A_0$ is an operator from the commutant $C_{H^r} = C_{H_1^{r_1} H_2^{r_2}}$, we suppose that its action on an arbitrary power $z^k$ has the form

$$Lz^k = \sum_{n=0}^{\infty} b_{k,n} z^n \qquad (15)$$

with unknown coefficients $b_{k,n}$. Then the expressions of $LH^r z^k$ and $H^r L z^k$ are

$$\begin{aligned} LH^r z^k &= L(\beta_{1,k}^{r_1} \beta_{2,k}^{r_2} z^k) = \\ &= \beta_{1,k}^{r_1} \beta_{2,k}^{r_2} L z^k = \beta_{1,k}^{r_1} \beta_{2,k}^{r_2} \sum_{n=0}^{\infty} b_{k,n} z^n \end{aligned} \qquad (16)$$

$$\begin{aligned} H^r L z^k &= H^r \left( \sum_{n=0}^{\infty} b_{k,n} z^n \right) = \\ &= \sum_{n=0}^{\infty} b_{k,n} H^r z^n = \sum_{n=0}^{\infty} b_{k,n} \beta_{1,n}^{r_1} \beta_{2,n}^{r_2} z^n. \end{aligned} \qquad (17)$$

If we equate the coefficients of the equal powers in (16) and (17), then

$$\beta_{1,k}^{r_1} \beta_{2,k}^{r_2} b_{k,n} = \beta_{1,n}^{r_1} \beta_{2,n}^{r_2} b_{k,n}, \quad n = 0,1,2,\ldots. \qquad (18)$$

Taking into account that $q_1 \neq q_2$ and the form of the coefficients in (13) and (14), we have that $\beta_{1,k}^{r_1} \beta_{2,k}^{r_2} \neq \beta_{1,n}^{r_1} \beta_{2,n}^{r_2}$ and then it follows that

$$b_{k,n} = \begin{cases} d_k \text{ - arbitrary}, & \text{for } n = k \\ 0, & \text{for } n \neq k \end{cases} \qquad (19)$$

This reduces the series in (15) to only one term:

$$Lz^k = d_k z^k, \quad d_k \text{ - arbitrary}. \qquad (20)$$

Finally, if an arbitrary analytic function $y \in A_0$ has a power series expansion $y(z) = \sum_{k=0}^{\infty} c_k z^k$ with coefficients $c_k = \frac{y^{(k)}(0)}{k!}$, then

$$Ly(z) = L\left( \sum_{k=0}^{\infty} c_k z^k \right) = \sum_{k=0}^{\infty} c_k L z^k = \sum_{k=0}^{\infty} c_k d_k z^k, \qquad (21)$$

which is the desired representation (12).

**Minimal commutativity**

First we have to describe the operators generated by $H^r$ and then this description will be compared with the one of the commutant $C_{H^r}$ given in Theorem 1.

**Theorem 2:**

Let us denote for simplicity of the writing the composition (14) and the coefficient in it by one letter

$$\begin{aligned} Hz^k &= H^r z^k = H_1^{r_1} H_2^{r_2} z^k = \beta_k z^k, \\ \beta_k &= \beta_{1,k}^{r_1} \beta_{2,k}^{r_2}. \end{aligned} \qquad (22)$$

Then the operators $Ay(z) = \sum_{j=0}^{\infty} a_j H^j y(z)$ generated by the operator $H: A_0 \to A_0$ have the form

$$Ay(z) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{y^{(k)}(0)}{k!} a_j \beta_k^j z^k. \qquad (23)$$

**Proof:** This follows immediately from the representation the action of the powers $H^j$ of $H$ on functions $y \in A_0$:

$$\begin{aligned} H^j y(z) &= H^j \left( \sum_{k=0}^{\infty} \frac{y^{(k)}(0)}{k!} z^k \right) = \\ &= \sum_{k=0}^{\infty} \frac{y^{(k)}(0)}{k!} H^j z^k = \sum_{k=0}^{\infty} \frac{y^{(k)}(0)}{k!} \beta_k^j z^k. \end{aligned} \qquad (24)$$

Let us make the definition of the minimal commutativity more precise. One can use two different variants of the definition, namely *finite* and *infinite* minimal commutativity

If in an algebra $K$ of operators the notion convergence of sequences $H_n \to H$ is defined and it is compatible with the algebraic operations at least so that $H_n \to H$ and $A_n \to A$ imply $HA_n \to HA$, $A_n L \to AL$, and $H_n + A_n \to H + A$, then the element $H$ is called (*infinitely*) minimally commutative if its commutant $C_H$ consists only of elements of the form

$$A = \sum_{j=0}^{\infty} a_j H^j = \lim_{n\to\infty} \sum_{j=0}^{n} a_j H^j = \lim_{n\to\infty} A_n, \quad a_j \in \boldsymbol{C}. \quad (25)$$

If the commutant $C_H$ contains only elements of the form

$$A_n = \sum_{j=0}^{n} a_j H^j, \ a_j \in \boldsymbol{C}, \quad (26)$$

with finite sum, then $H$ is called *finitely* minimally commutative.

**Theorem 3.**
If the operator $H$ defined by (22) is considered in the subspace $S \subset A_0$ of the polynomials, then it is finitely minimally commutative.

**Proof:** Let $y(z) = \sum_{k=0}^{r} c_k z^k \in S$ be an arbitrary polynomial. Then by (6) any operator $L$ from the (finite) commutant $C_H$ in $S$ must have a polynomial form

$$Ly(z) = \sum_{k=0}^{q} c_k d_k z^k \quad (27)$$

with zero coefficients $c_k$ of the highest degrees if $q > r$.

From (26) the action of an arbitrary operator $A \in G_H$ generated by $H$ on $y$ is

$$\begin{aligned} Ay(z) &= \sum_{j=0}^{n} a_j H^j y(z) = \sum_{j=0}^{n} a_j \sum_{k=0}^{q} c_k H^j z^k = \\ &= \sum_{j=0}^{n} a_j \sum_{k=0}^{q} c_k b_k^j z^k = \sum_{k=0}^{q} c_k \sum_{j=0}^{n} a_j b_k^j z^k, \end{aligned} \quad (28)$$

i.e. $Ay(z)$ is also a polynomial of degree at most $q$.
Now, equating the coefficients of the equal powers in (27) and (28), we have to solve the linear system with unknowns $a_j$:

$$\sum_{j=0}^{n} a_j b_k^j = d_k, \qquad k = 0,1,2,\ldots,q. \quad (29)$$

We can suppose that $n = q$ since if $n > q$, we can take $a_{q+1} = \ldots = a_n = 0$ and the system (29) becomes with equal number of equations and unknowns:

$$\sum_{j=0}^{q} b_k^j a_j = d_k, \qquad k = 0,1,2,\ldots,q. \quad (30)$$

The determinant of the system is the non-vanishing Wandermonde's one

$$W(b_0, b_1, \ldots, b_q) = \begin{vmatrix} 1 & b_0 & \cdots & b_0^q \\ 1 & b_1 & \cdots & b_1^q \\ \vdots & \vdots & \ddots & \vdots \\ 1 & b_q & \cdots & b_q^q \end{vmatrix} = \prod_{0\le\mu<\nu\le q} (b_\mu - b_\nu) \neq 0, \quad (31)$$

since $b_\mu \neq b_\nu$ for $\mu \neq \nu$ by the definition of $H$. Hence the system (30) has an unique solution $(a_0, a_1, \ldots, a_q)$ and the operator $H$ is finitely minimally commutative.

**Remark:**
We proved Theorem 2 for the subspace $S \in A_0$ of the polynomials. If we consider the whole space $A_0$, then it is natural to try to prove the infinite minimal commutativity of $H$, but then a linear system like (29) with infinitely many equations and infinitely many unknowns has to be solved. We cannot give a positive or negative result, but it is at least clear that a representation of the operators of the commutant with finite sum as (26) is impossible in the general case when infinitely many $d_k$ are chosen different from zero. Indeed, if we suppose that the operator $H : A_0 \to A_0$ is finitely minimally commutative, then

$$\sum_{j=0}^{q} b_k^j a_j = d_k, \qquad k = 0,1,2,\ldots. \quad (32)$$

The first $q+1$ equations have an unique solution $(a_0, a_1, \ldots, a_q)$ depending on $d_0, d_1, \ldots, d_q$ as in Theorem 2. But from the next equations for $k \geq q+1$ we see that

$$d_k = \sum_{j=0}^{q} b_k^j a_j (d_0, d_1, \ldots, d_q), \quad (17)$$

i.e. all $d_k$, $k \geq q+1$, depend on the first $d_k$, $0 \leq k \leq q$, and cannot be arbitrarily chosen. This contradicts the description of the commutant $C_H$.

## The case of increasing the powers

Due to the lack of space we are not able to consider here the case of *increasing* the powers by the operators in the composition (9) and also in the mixed cases. Let us only mention that the result which will be proved in **Part II** (Hristova 2013d) of this paper states that

*The composition $H^r = H_1^{r_1} H_2^{r_2}$ is minimally commutative if and only if the total change of the powers is exactly one.*

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