THE EQUILIBRIUM OF A BODY LOADED WITH A SPATIAL SYSTEM OF FORCES

Asen Stoyanov

University of Mining and Geology "St. Ivan Rilski", 1700 Sofia

ABSTRACT. A comparative research has been carried out on the equilibrium of a body loaded with a spatial system of forces from which one of the distributed loads has an intensity which is changed after a non-linear law. For the purposes of this article, two analytical solutions to a specific task have been compared. The manual solution is classic. In it, the resultant forces and resultant moment from the distributed loads are determined. After that, the concentrated forces are decomposed into components, and the equations for equilibrium are composed. Finally, the unknown values are determined and the solution is checked.

The second solution is performed by means of MathCAD 15. The graph of the non-linear function $q_2(y)$ is automatically depicted in the figure. The equations for the equilibrium are represented in the $A.X = B$ matrix form, and its solution is the solution to the problem.

Keywords: three-dimensional system of forces, inverse matrix, MathCAD

РАВНОВЕСИЕ НА ТЯЛО, НАТОВАРЕНО С ПРОСТРАНСТВЕНА СИСТЕМА ОТ СИЛИ

Асен Стоянов

Минно-геоложки университет "Св. Иван Рилски", 1700 София

РЕЗЮМЕ. Проведено е сравнително изследване на равновесието на тяло, натоварено с пространствена система от сили, от която едно от разпределените натоварвания има интензивност, която се променя по нелинеен закон. В статията за конкретна задача са сравнени две аналитични решения. Ръчното решение е класическо. В него се определят равнодействащите сили и резултантният момент от разпределените натоварвания. След това концентрираните сили се разлагат на компоненти и се съставят уравненията за равновесие. Накрая се определят неизвестните и се проверява решението.

Второто решение се изпълнява с MathCAD 15. Графиката на нелинейната функция се изобразява автоматично на дадената фигура. Уравненията за равновесие са представени в матрична форма $A.X = B$, и нейното решение е решение на задачата.

Ключови думи: пространствена система от сили, обратна матрица, Маткад

Introduction

The article studies the equilibrium of a beam with a broken axis loaded with a spatial system of forces. One of the two distributed loads is with variable intensity. The function that describes the change of this intensity is square.

Determining the resultant force $\vec{R}_2$ and the resultant moment $\vec{M}_{2x}$ requires integration within the boundaries of the section loaded with $q_2(y)$. The resulting algebraic projection M_2x is directly involved in the equilibrium equation (for this specific example, in equation $\sum Mx_i = 0;$).

Difficulties that arise when solving problems in theoretical mechanics, and in particular in statics, are mathematical. In the example under consideration, integrating a square function is not a problem. However, if the intensity is expressed by a function other than a polynomial, the difficulties become very prominent.

Such problems in engineering practice are not uncommon. Their solution is easy when using any of the mathematical packages, such as MATLAB, Maple, MathCAD, and the like.

The beam is studied classically, “by hand”, and with the help of the MathCAD package. The analysis of the two types of solution makes it easy to assess their efficiency.

A similar problem has been solved by Doev and Dronin (2016). The authors cited have chosen a positive function for the intensity in the loaded section.

In the current article, the solution to similar a problem has been improved. The author has chosen the law for the change of intensity in such a manner that the distributed load changes its direction of action over part of the loaded section - see fig. 1.

The equilibrium of $3D$ systems of forces is examined also by Bertyaev (2005) and Stoyanov (2014, 2016).

The system of forces acting on a free moving body is successfully studied in a dynamical setting and by means of the MATLAB programme (Ivanov, A., 2014, Ivanov, I., Y. Yavorova, 2017).

Solution "by hand"

The beam is stationary in the thee-dimensional space within the reading system $Oxyz$ (see fig. 1).

The support devices at points A and B have been replaced by the corresponding reaction forces (see fig.1).

The assignment is to find analytically all the reaction forces, if the geometry and load on the beam are known:

$$a = 3m;\ b = 4m;\ c = 3m;\ d = 2{,}5m;\ \alpha = \frac{\pi}{3};$$

$$\beta = \frac{\pi}{4};\ P_1 = 35kN;\ P_2 = 32kN;$$

$$q_1 = 33kN/m;$$

$$q_2(y) = -50.y^2 + 50.y + 287{,}5\,kN/m.$$

Solution:

1) *Decomposition of the forces $\overrightarrow{P_1}$ and $\overrightarrow{P_2}$ into components*

$$\overrightarrow{P_1}\left| \begin{aligned} P_{1y} &= P_1.\sin\alpha = 30{,}311kN; \\ P_{1z} &= P_1.\cos\alpha = 17{,}5kN; \end{aligned} \right.$$

$$\overrightarrow{P_2}\left| \begin{aligned} P_{2x} &= P_2.\sin\beta = 22{,}63kN; \\ P_{2y} &= P_2.\cos\beta = 22{,}63kN. \end{aligned} \right.$$

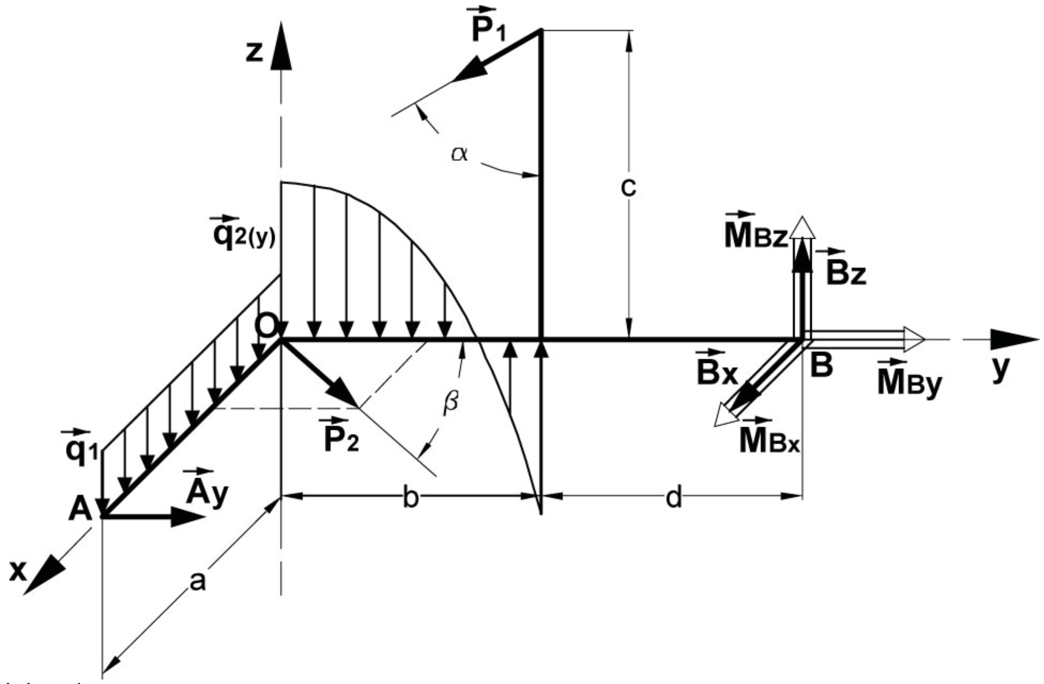


Fig. 1. Calculation scheme

2) *Determining the sizes of the resultants $\vec{R}_1$ and $\vec{R}_2$*

$$R_1 = q_1.a = 33.3 = 99kN;$$

$$R_2 = \int_0^b q_2(y).dy = \int_0^b (-50.y^2 + 50.y + 287{,}5).dy;$$

$$R_2 = -16{,}67.\,y^3\Big|_0^4 + 25.\,y^2\Big|_0^4 + 287{,}5.\,y\Big|_0^4$$

$$R_2 = 483{,}333\,kN.$$

3) *Determining the sizes of the resultant moment $\overrightarrow{M}_{2x}$*

$$M_{2x} = \int_0^b y.q_2(y).dy;$$

$$M_{2x} = -12{,}5.y^4\Big|_0^4 + 16{,}667.\,y^3\Big|_0^4 + 143{,}75.\,y^2\Big|_0^4;$$

$$M_{2x} = 166{,}667\,kN.m.$$

4) *Determining the coordinate of the resultant force $\overrightarrow{R_2}$*

$$y_2 = \frac{M_{2x}}{R_2};\quad y_2 \approx 0{,}345m.$$

5) *Compiling a system of equations for the equilibrium of the beam*

$$\sum X_i = 0;\ B_x + P_{2x} = 0;$$

$$\sum Y_i = 0;\ A_y + P_{2y} - P_{1y} = 0;$$

$$\sum Z_i = 0; -R_1 - R_2 - P_{1z} + B_z = 0;$$
$$\sum Mx_i = 0;$$
$$P_{1y}.c - P_{1z}.b + M_{Bx} - M_{2x} + B_z.(b+d) = 0;$$

$$\sum M_{iy} = 0;\ R_1.a.0{,}5 + M_{By} = 0;$$

$$\sum M_{iz} = 0;\ A_y.a + M_{Bz} - B_x.(b+d) = 0.$$

6) *Solving the equilibrium equations*

The equilibrium equations in this case (p.5)) are independent with relation to the unknown values and can be solved separately.

$$B_x = -22{,}63\,kN;$$

$$A_y = -22{,}63 + 30{,}311 = 7{,}681\,kN;$$

$$B_z = 99 + 483{,}12 + 17{,}5 = 599{,}833\,kN;$$

$$M_{Bx} = -30{,}311.3 + 17{,}5.4 + 166{,}667 - 6{,}5.599{,}62$$
$$M_{Bx} = -3753{,}1805\,kN.m;$$

$$M_{By} = -99.3.0{,}5 = -148{,}5\,kN.m;$$

$$M_{Bz} = -7{,}681.3 - 22{,}63.6{,}5 = -170{,}138\,kN.m.$$

If it is necessary to solve a linear system of six equations with six unknown values "by hand", the system can be presented in the compact matrix form –

$$A.S = B \qquad (1)$$

Where:

- A is the matrix of coefficients in front of the unknowns with dimensionality NxN ($N = 6$);
- S is a vector whose elements are unknown reaction forces and reaction moments;
- B is a vector whose elements are known magnitudes (the free members of the system on p. 5) multiplied by (-1).

The solution to matrix equation (1) is searched in the species $S = A^{-1}.B$. In order to determine the reversibility of A (i.e. $|A| \neq 0$), it is necessary to use the Gauss-Jordan method.

7) *Verification*

The head moment of the forces (active and passive) that are applied to the beam relative to axis "s" with a single vector $\vec{e}_s = \frac{1}{\sqrt{3}}(\vec{i} + \vec{j} + \vec{k})$ must be equal to zero –

$$M_s = \frac{1}{\sqrt{3}}(R_1.0{,}5.a - R_2.0{,}345 + A_y.a -$$
$$- P_1.\cos\alpha.b + P_1.\sin\alpha.c + B_z.(b+d) -$$
$$- B_x.(b+d) + M_{Bx} + M_{By} + M_{Bz}) \overset{?}{=} 0;$$

$$Ms = \frac{1}{\sqrt{3}}(148{,}5 - 166{,}75 + 23{,}043 - 70 +$$
$$+ 90{,}933 + 3898{,}9145 + 147{,}095 -$$
$$- 3753{,}1805 - 148{,}5 - 170{,}138 =$$
$$= \frac{1}{\sqrt{3}}.(4308{,}4855 - 4308{,}5685) =$$
$$= -0{,}04792 \approx 0!$$

Solution to the problem with the MathCAD package

The algorithm of the solution is as follows:

- The output data are introduced – see fig.2;
- The resultant forces $\vec{R}_1$ and $\vec{R}_2$, the resultant moment $\overrightarrow{M_{2x}}$, and the "y_2" coordinate are determined – see fig.2.;
- The distributed load $q2(y)$ is graphically presented – see fig. 2;
- The vectors $p_1,\ p_2,\ R_1,\ and\ R_2$ are formed - see fig.2.;
- The square matrix is formed and its reversibility is verified, i.e. $\det A \neq 0$ – see fig.2;
- Vector B is formed with elements that are free members in the equations from p.5) and those are multiplied by (-1) – see fig. 2.
- The support reactions are determined – see fig. 2.

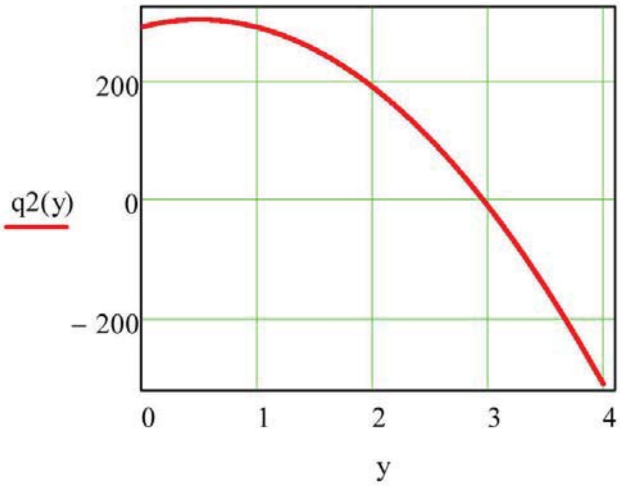
$$P1 := 35 \quad P2 := 32 \quad q1 := 33 \quad q2(y) := -50 \cdot y^2 + 50 \cdot y + 287.5$$

$$a := 3 \quad b := 4 \quad c := 3 \quad d := 2.5 \quad \alpha := \frac{\pi}{3} \quad \beta := \frac{\pi}{4} \quad \mathrm{ORIGIN} := 1$$

$$R2 := \int_0^b q2(y)\,dy \quad M2x := \int_0^b y \cdot q2(y)\,dy \quad y2 := \frac{M2x}{R2} \quad R2 = 483.333$$

$$M2x = 166.667 \quad y2 = 0.345 \quad y := 0, .0001..4 \quad p1 := (0 \;\; -P1 \cdot \sin(\alpha) \;\; -P1 \cdot \cos(\alpha))^T$$

$$R1 := (0 \;\; 0 \;\; -99)^T \quad R2 := (0 \;\; 0 \;\; -483.333)^T \quad p2 := (P2 \cdot \sin(\beta) \;\; P2 \cdot \cos(\beta) \;\; 0)^T$$



$$A := \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & b+d & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ a & -(b+d) & 0 & 0 & 0 & 1 \end{bmatrix} \quad |A| = -1 \quad B := \begin{bmatrix} -p2_1 \\ -p2_2 - p1_2 \\ (-R1)_3 - R2_3 - p1_3 \\ -\left(-p1_2\right) \cdot c + \left(-p1_3\right) \cdot b + M2x \\ R1_3 \cdot .5 \cdot a \\ 0 \end{bmatrix}$$

$$S := (\,Ay \;\; Bx \;\; Bz \;\; Mx \;\; My \;\; Mz\,)^T \quad S := A^{-1} \cdot B \quad S = \begin{pmatrix} 7.683 \\ -22.627 \\ 599.833 \\ -3.753 \times 10^3 \\ -148.5 \\ -170.129 \end{pmatrix}$$

Fig. 2. Partially automated solution to the beam with the MathCAD package

Conclusion

The actual directions of reaction $\vec{B}_x$ and reactive moments $\vec{M}_x, \vec{M}_y$ *and* $\vec{M}_z$ are opposite to the displayed ones - see fig.1.

The study presented, in which the problem is solved both "by hand" and with the MathCAD package for mathematical research, gives a clear idea of the advantages of the MathCAD application compared to the solution "by hand".

The solution "by hand" is sometimes accompanied not only by the difficulties mentioned in the introduction, but also by errors. The latter are difficult to detect because the process of tracking the solution is longer than that with the MathCAD package. Furthermore, when the problem is solved correctly, it is possible for the routine error to be made in the course of the verification.

When it a linear system of six equations with six unknowns is solved "by hand", the Gauss-Jordan method must be applied correctly, i.e. $(A|E \rightarrow E|A^{-1})$. The plausible presentation of the distributed load $\vec{q}_2(y)$ on the diagram by hand (see fig.1.) requires the use of tools for drawing (the calculation scheme in the fig. 1. is drawn with AutoCAD).

The partially automated solution to the beam with the MathCAD package is quick and compact and it accurately represents the square function $q_2(y)$ – see fig. 2.

The solution to the problem with the MathCAD package cannot guarantee the lack of errors, but those can easily be found in the short and clear record – see fig. 2.

The use of the graphic editor in MathCAD package helps for establishing the connection between a geometric or a force parameter and reaction forces and reaction moments.

Acknowledgements:
The author wishes to thank his colleague L. Georgiev, who read the material and made valuable remarks that have improved its outer appearance.

References

Бертяев, В. Теоретическая механика на базе MathCAD практикум, Санкт–Петербург, „БХВ–Петербург", 2005, 739 с. (Bertyaev, V. Teoreticheskaya mehanika na baze MathCAD praktikum, Sankt-Peterburg, "BHV-Peterburg", 2005, 739 p.)

Доев, В.С., Ф.А. Доронин. Сборник заданий по теоретической механике на базе MathCAD, Санкт Петербург, Москва, Краснодар, Лань, 2016, 585 с. (Doev, V.S., F.A. Doronin, Sbornik zadaniy po teoreticheskoy mehanike na baze MathCAD, Sankt-Peterburg, M., Krasnodar, Lany, 2016, 585p.)

Иванов А., Пространствено изследване летежа на тенис топка, сп. Механика на машините, том 2, год. XII, 2014, 34 ÷ 37 с. (Ivanov A., Prostranstveno izsledvane letezha na tenis topka, sp. Mehanika na mashinite, tom 2, god. XII, 2014, 34 ÷ 37p.)

Стоянов, А. Определяне равновесието на пространствена система сили посредством MathCAD, XIV Международна научна конференция ВСУ'2014, 2014г. 44–47 с. (Stoyanov, A., Opredelyane ravnovesieto na prostranstvena sistema sili posredstvom MathCAD, XIV Mezhdunarodna nauchna konferentsiya VSU'2014, 44–47)

Стоянов, А., Матрични операции с MathCAD в теоретичната механика Статика, Кинезиология БГ, София 2016, 165с. (Stoyanov, A. Matrichni operatsii s MathCAD v teoretichnata mehanika Statica, S. Kineziologiya BG, 2016, 165p.)

Ivanov A., Javorova J., Three dimensional golf ball flight, J. Tehnomus, P-ISNN-1224-029X, E-ISNN-2247-6016, 2017, 54 ÷ 61 p.

The article is reviewed by Prof. Dr. Mihail Valkov and Assoc. Prof. Dr. A. Ivanov.