FROM RHEOLOGY TO PLASTICITY AND VISCOPLASTICITY

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ABSTRACT. Obtaining some models of non-linear behaviour of the materials have 2 directions: the study of rheological properties and define of the form of the equations for a three-dimensional solicitation. The rheology, the science that study the matter in time, from the point of view of the flow, of the deformations, allows obtaining some correlations between stress, deformations and their derivates, and characterize the nature of the components. We will introduce in this paper the most complex behaviours, starting from elementary notions and the description of the different criterias which allow the generalization of the obtained equations in three-dimensional cases.
Keywords: elasticity, plasticity, viscoplasticity, rheological model, behaviour, criterion, hardening.

ОТ РЕОЛОГИЯ ДО ПЛАСТИЧНОСТ И ВИСКОЗОПЛАСТИЧНОСТ
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РЕЗЮМЕ. Получените модели на нелинейно поведение на материалите имат две посоки: изследване на реологичните свойства и определяне вида на уравненията за три-измерни системи. Реологията е наука, занимаваща се с проблемите за изясняване на връзката между напреженията и скоростта на деформация на различните видове среди (течни и твърди), което позволява получаването на някои корелации между напрежение, деформации и техните производни, и характеризира природата на компонентите. В този доклад са разгледани най–сложните поведения, започвайки от най-елементарните, като са описани различни критерии, които позволяват да се обобщят получените уравнения в три-измерни случаи.
Ключови думи: еластичност, пластичност, вискозитет, реологичен модел, поведение, критерий, втвърдяване

1. Fundamental elementary notions

The qualitative form of the materials behaviour result after realizing some simple tests, allow them to be framed in well – defined classes. The fundamental behaviour that could be represented through elementary mechanical systems are: the elasticity, the plasticity and the viscoplasticity. The most well-known elements are, fig1:

- the resort, which symbolizes the linear elasticity for which the deformation is reversible and it exists a relation between the charging parameters and the deformation ones (fig.1.a);
- the damper, which schematize the linear viscosity (fig.1.b) or non-linear (fig.1.c). The viscosity is pure if there is a relation between loading and the speed of this; if this relation is linear the model is related to Newton's law;
- the patina, which describe the appearance of the permanent deformation if the loading is big enough (fig.1.d). If the first step of the permanent deformation does not evaluate with the loading, the behaviour is perfect plastic and moreover, if the deformation between the flow is neglected, the model is rigid – perfect plastic.

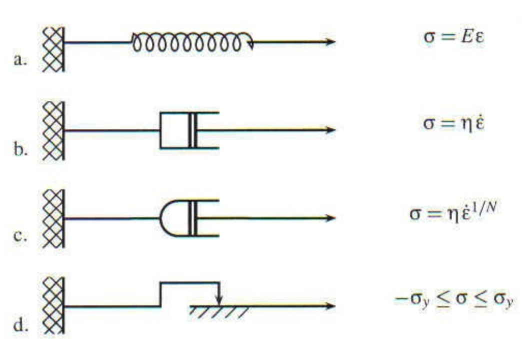

Fig. 1. Fundamental notions in the representation of behaviour

These elements could be combined, making rheological models, which represent mechanical systems, used as a support in defining the models.

The response of these systems could be thought in 3 different plans, which allow showing the obtained behaviour by a certain type of experiments:

- hardening or monotone increase of the strain (strain – stress plan, ε - σ);
- creep or constant loading (time – stress plan, t - σ);
- relaxation or constant strain (time – strain plan, t - ε).

2. Uniaxial plasticity

The association between a resort and a patina in series produces a elastic perfect plastic behaviour (fig.2.a) the system not being able to support a stress which's absolute value is bigger then σ_y.

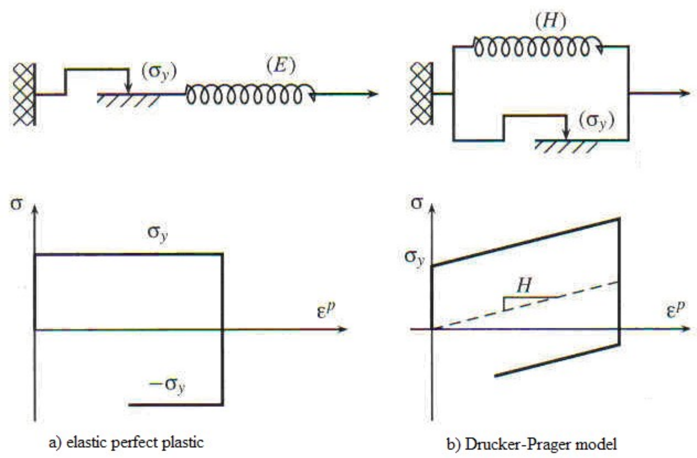

Fig. 2. Association in series and parallel of the patina and resort

The characterization of this model is made by considering the loading function f, dependent of the only variable σ, defined by:

$$f(\sigma) = |\sigma| - \sigma_y \tag{1}$$

The elasticity field belongs the f's negative values and the system' behaviour resume itself to the following equations:

- the elasticity field, if f < 0

$$(\dot{\varepsilon} = \dot{\varepsilon}^e = \dot{\sigma}/E)$$

- the elastic unloading, if f = 0 and

$$\dot{f} < 0 \quad (\dot{\varepsilon} = \dot{\varepsilon}^e = \dot{\sigma}/E)$$

- plastic flow if f = 0 and

$$\dot{f} = 0 \quad (\dot{\varepsilon} = \dot{\varepsilon}^p)$$

In elastic domain, the plastic strain rate $\dot{\varepsilon}^p = 0$, the elastic strain rate becoming zero, during the plastic flow. The model is without hardening, because the stress's level varies at the end of the elastic field. The model is susceptible to reach infinite deformations under a constant loading, leading to the damage of the system by excessive deformation. The association in parallel, fig.2.b - Prager's model, of these 2 elements is related with a behaviour in which the hardening is present; it is a linear hardening and is named kinematical, because it depends of the actual value of the plastic strain. In this case, the loading function depends of the actual value of the plastic strain. So, this function depends of the applied stress and the intern stress, X, that characterize the new neutral state of the material:

$$f(\sigma) = |\sigma - X| - \sigma_y \tag{2}$$

The stresses evaluate during the plastic flow, being useful as control variables. It always exists the possibility of expressing the plastic strain rate according to the rate of the total strain:

$$\dot{\varepsilon}^p = \frac{E}{E + H} \cdot \dot{\varepsilon} \tag{3}$$

It is remarkable to notice that the calculus of the dissipated energy during a cycle, produces the same result as the first scheme, which indicates the fact that for his type of behaviour a part of the energy is temporarily stocked in the material (here is the resort) and wholly restituted at downloading. This gives a physical illustration of the reversible hardening notion, when other rules of cinematic non-linear hardening are accompanied by a dissipation of this energy.

In uniaxial elastoplasticity, the loading – unloading conditions are expressed in general case through:

- the elasticity field if

$$f(\sigma, A_i) < 0 \qquad (\dot{\varepsilon} = \dot{\sigma}/E)$$

- elastic unloading if

$$f(\sigma, A_i) = 0 \quad \text{and} \quad \dot{f}(\sigma, A_i) = 0 \quad (\dot{\varepsilon} = \dot{\sigma}/E)$$

- plastic flow if

$$f(\sigma, A_i) = 0 \quad \text{and} \quad \dot{f}(\sigma, A_i) = 0 \quad (\dot{\varepsilon} = \dot{\sigma}/E + \dot{\varepsilon}^p)$$

In the general case, the H model depends on the strain and / or the hardening variables; the value of the plastic model in the point (σ, A_i) is obtained by writing that the representative point at the loading during the flow remains on the limit of the elasticity field, and the resulted equation is named "the coherence equation":

$$\overset{\circ}{f}(\sigma, A_i) = 0 \qquad (4)$$

In these examples, the elasticity field is either fix or mobile, its length being conserved. The first case does not need any hardening variable, in the second, the X variable occurs and its depend on the actual value of the plastic strain which on the general case will become a tensorial variable. The type of hardening which is related to it is the cinematic hardening (fig.3.b). In the particular case illustrated by the rheological model, the evolution of the X variable is linear according to the plastic strain, this being the model of linear kinematic hardening (Prager, 1958).

Another elementary evolution of which the elasticity field could support is the expansion (fig.3.a), related to a material of which's elasticity field records a growth in length, but it remains centrated in the origin; it is about an isotropic hardening (Taylor and Quinney, 1931), in the f function, the variable R which occurs, is the dimension of the elasticity field:

$$f(\sigma, X, R) = |\sigma| - R - \sigma_y \qquad (5)$$

The evolution of this variable is the same, no mather of the sign of the cumulated variation of plastic strain, p, a variable which's derivate is equal with the absolute value of the plastic strain rate, $\overset{\circ}{p} = \left|\overset{\circ}{\varepsilon}{}^{p}\right|$. So, it does not existing a difference between p and ε^p while the loading is monotone increase. In this case, the verification of the condition means expressing the fact that the actual value of the stress on the bound of the elasticity field:

- for kinematic hardening: $\sigma = X + \sigma_y$
- for izotropical hardening: $\sigma = R + \sigma_y$

which means the fact that the evolution law of the hardening variable is the one that determines directly the form of the extension's curve.

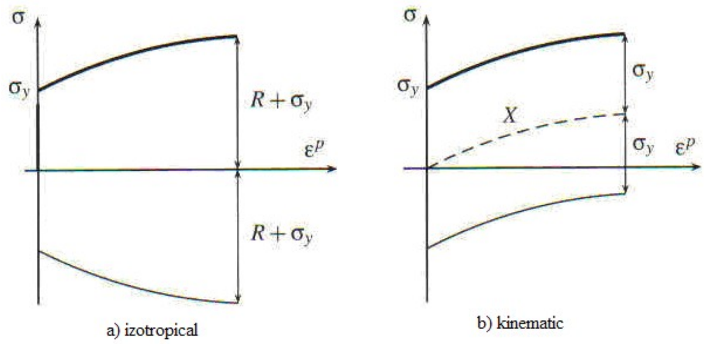

Fig. 3. Representation of the two types of hardening

The izotropical hardening is mostly used for important deformations (over 10 %). The kinematic hardening continuous to play an important role after the unload, even for big deformations and it is prevalent for small deformations and the cycle loadings, allowing the correct simulation of Bauschinger effect, meaning the fact that the elasticity stress in compression unloads related to the initial stress as a following of a pre-hardening in extension.

3. Viscoelasticity and viscoplasticity

Viscoelasticity could be well defined through simple models Maxwell and Voigt which group a damper in series and in parallel (fig.4) or through the utilization of some composed models, such as Kelvin – Voigt or Zener (fig.5). The particularity of Voigt model is that it does not present instant elasticity, its function of relaxation not being continuous and derivable on pieces.

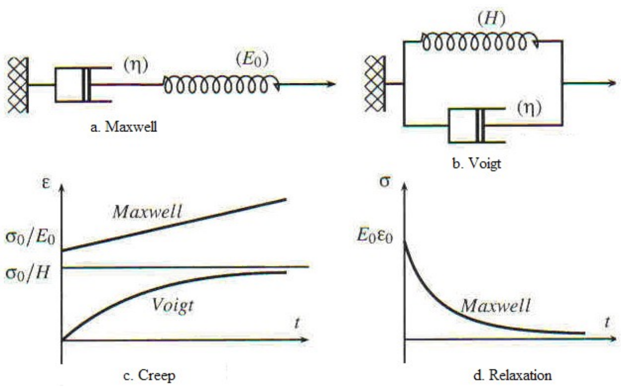

Fig. 4. Representation of the simple models Maxwell and Voigt

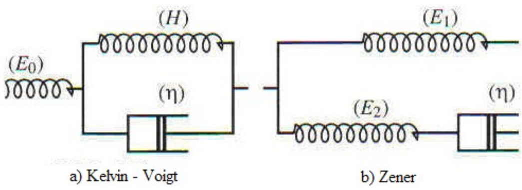

Fig. 5. Examples of composed models

The Voigt model is not used for relaxation, unless putting it under deformation is progressive and because of that, in order to effectuate the calculus of structure, it was associated with a series resort – the Kelvin – Voigt model. Under the effect of a stress σ_0 = const. in time, the deformation goes asymptotic to σ_0 / H, meaning that the creep is limitated. In the case of

Maxwell model the creep's rate is constant and the disappearance of the stress during an experiment is total. By adding a simple damper to a simple model, there is the possibility to pass easily from a model which has a plastic behaviour, independent from time, to a viscoplastic model (fig.6), the resulted model being the generalized Bingham model. By eliminating the resort in series the viscoplastic rigid model is obtained, and by suppressing the resort in parallel, there will be no hardening.

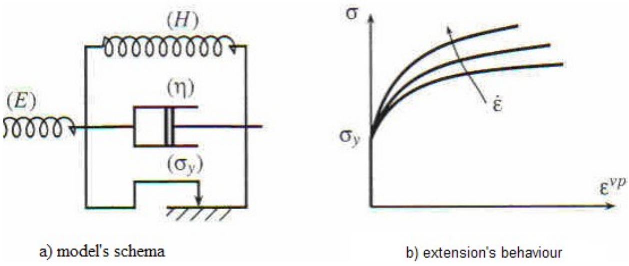

Fig. 6. Bingham generalized model

In the case of a viscoplastic model, there are 2 possibilities to introduce the hardening, by conserving the possibilities to action either upon the plastic variables – the case of the models with additive hardening, or upon the viscous stresses, when we talk about the models with a multiple hardening, a representative law describing this type of hardening being Lemaitre's law:

$$\overset{\circ}{\varepsilon}^{vp} = \left(\frac{|\sigma| K}{}\right)^{n} \mathrm{sign}(\sigma)$$

- Norton's law (6)

$$\overset{\circ}{\varepsilon}^{vp} = \left(\frac{|\sigma| K}{}\right)^{n} p^{-n/m} \mathrm{sign}(\sigma) \quad , \quad \text{with} \quad \overset{\circ}{p} = \left|\overset{\circ}{\varepsilon}^{vp}\right|$$

- Lemaitre's law (7)

where K, n, m – the material's coefficients.

4. Criterias

The used models, offer a uniaxial loading, shown an elasticity field, in the stresses domain and the hardening variables, for which it does not exist plastic flow or viscoplastic. The trace of this field on the stress axe is limited at a segment which can support a translation or an extension, sometimes even limited to a point. On the other side, certain models are capable to represent a maximum stress, supported by the material. The main classes of criterias in writing the model are:

a- criterias in which the hydrostatic pressure does not appear (the new criterion von Mises and Tresca);
b- criterias which take into account the hydrostatic pressure (the criterion Drucker – Prager, Mohr – Coulomb, the "closed" criterias);
c- anizotropical criterias.

4.1. Criterias in which the hydrostatic pressure does not appear

While the trace on the stress' tensor does not appear, the most simple criterion is the one that used only the second invariant of the stress' tensor of J, which is related to an ellipse in the space of the symmetric tensors, meaning the von Mises criterion:

$$f(\underset{\sim}{\sigma}) = J - \sigma_y \qquad (8)$$

σ_y – the elasticity limit in extension.

This makes the maximum shears to appear in every main plan, represented through the quantities (σ_i - σ_j). Specific to Tresca criterion is not to keep from these quantities, only the highest values. Adding a pressure to each term of the diagonal does not modify the criterion's value. The expression, contrary to the von Mises criterion, does not define a regular surface (the discontinuity of the normal, angular points):

$$f(\underset{\sim}{\sigma}) = \max_{i,j}\left|\sigma_i - \sigma_j\right| - \sigma_y \qquad (9)$$

It is interesting to compare these two criterias. Because being situated in the space of 6 (or 9) components of the stress' tensor is not an issue, we must see the boundaries of the elasticity field in the subspaces with 2 and 3 dimensions. The representations are being made:
a) in extension – shear plan (fig.7.a) the only components σ = σ_{11} and $\tau = \sigma_{12}$ not being zero. The expressions of the criterias are reduced to:

- von Mises: $$f(\sigma, \tau) = \sqrt{\sigma^2 + 3\tau^2} - \sigma_y \qquad (10)$$

- Tresca: $$f(\sigma, \tau) = \sqrt{\sigma^2 + 4\tau^2} - \sigma_y \qquad (11)$$

b) in the main stresses' plan (σ_1, σ_2) (fig.7.b) when the stress σ_3 = 0:
- von Mises:

$$f(\sigma_1, \sigma_2) = \sqrt{\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2} - \sigma_y \qquad (12)$$

- Tresca:

$$f(\sigma_1, \sigma_2) = \begin{cases} \sigma_2 - \sigma_y & \text{if} \quad 0 \le \sigma_1 \le \sigma_2 \\ \sigma_1 - \sigma_y & \text{if} \quad 0 \le \sigma_2 \le \sigma_1 \\ \sigma_1 - \sigma_2 - \sigma_y & \text{if} \quad \sigma_2 \le 0 \le \sigma_1 \end{cases}$$

(13)

In a deviatoric plan, the criterion von Mises is represented through a circle, which is related to its interpretation, through octaedrical shear, the Tresca criterion is represented through a hexagon.

c) in the space of the main stresses each of these criterias of the main stresses each of these criterias is represented through a generating set cylinder (1, 1, 1) in the base of the defined curves in a deviatoric plan.

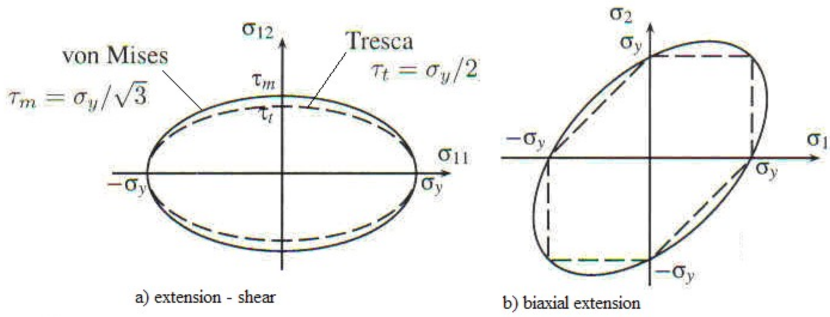

Fig. 7. Comparison of the Tresca and von Mises models

4.2. Criterias in which hydrostatic pressure is take into account

These criterias are necessary to represent the plastic deformation of the materials, lands or of the presence of fissures, the discontinuities of the materials, expressing the fact that a hydrostatic stress of compression it opposes to the plastic deformation. One of the consequences of their formulation is that they introduce a non-symmetry extension – compression.

The criterion Drucker – Prager is an expansion of the von Mises criterion, a linear combination between the second invariant and the trace of the stresses' in a deviatoric plan, being a circle:

$$f(\underset{\sim}{\sigma}) = (1-\alpha)\,J + \alpha\,I - \sigma_y \tag{14}$$

The limit of elasticity in extension remains σ_y and in compression is - σ_y / (1 – 2 α), α being a coefficient related to the material, α = 0 – 0.5 ($\alpha = 0 \Rightarrow$ von Mises criterion (fig.8).

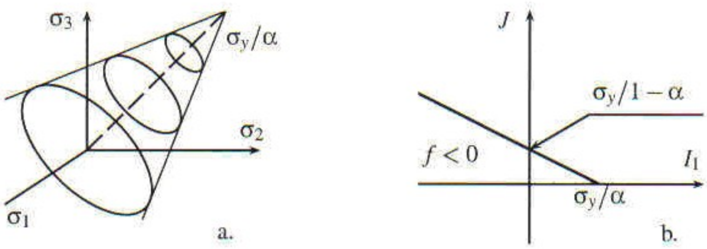

Fig. 8. The representation of Drucker – Prager criterion: a) in the main stress space; b) in the I_1 – J plan

The Mohr – Coulomb criterion has a certain resemblance with Tresca criterion, making the maximum shear to appear, but in the meantime the average shear, represented through the Mohr's center circle correspondent to the maximum shear:

$$f(\underset{\sim}{\sigma}) = \sigma_1 - \sigma_3 + (\sigma_1 + \sigma_3)\sin\varphi - 2\,C\cos\varphi \qquad \text{with} \quad \sigma_3 \le \sigma_2 \le \sigma_1 \tag{14}$$

This criterion assumes that the maximum shear that the material can support (T_t, fig.9.a) is as bigger as the normal stress compression is higher. The admitted limit is an intrinsic curve in the Mohr plan:

$$|T_t| < -\tan(\varphi)\,T_n + C \tag{15}$$

where: C – cohesion; φ - the internal friction angle of the material; $\varphi = 0$, $C \neq 0$ – pulverulent material; $\varphi \neq 0$, $C = 0$ – pure cohesive material.

In a deviatoric plan (fig.9.b) is obtained a non-regular characterized through the values:

$$\sigma_t = \frac{2\sqrt{6}\,(C\cos\varphi - p\sin\varphi)}{3+\sin\varphi}$$

$$\sigma_c = \frac{2\sqrt{6}\,(-C\cos\varphi + p\sin\varphi)}{3-\sin\varphi} \quad , \quad p = -\frac{1}{3} I_1 \tag{16}$$

These two criterias show the fact that the material becomes infinite resistant in triaxial compression, behaviour whose not generally verified for the real material sensible to hydrostatic pressure. In order to simulate on, for example the compaction, is reduced to "closed models" in which the limit curve is define through two pieces. As an example is the Cam-Clay model used for clays which's limit curve is defined by two ellipses in the plan (I_1 – J) or the "cap mode" model, which closes with an ellipse the criterion Drucker – Prager.

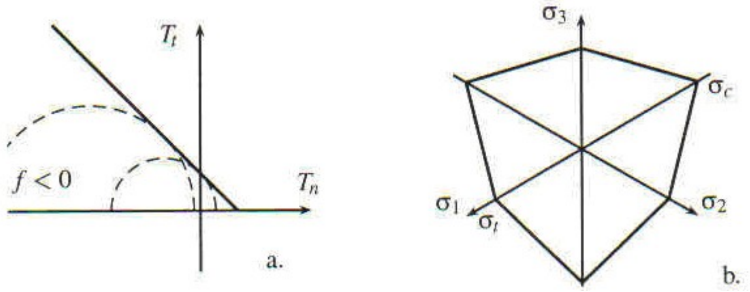

Fig. 9. Mohr – Coulomb criterion: a) in the Mohr plan; b) in deviatoric plan

4.3. Anizotropical criterias

If the loaded surface of a metallic material is measured experimental, it is seen that in the presence of unelastic deformations it records an extension, a translation and a distortion, the first two modifications being represented by the izotropical and kinematic hardenings, the last one not being considered by the current models.

There are anizotropical materials, such as composites materials. There are lots of possibilities of expansion of the izotropical criterias, in order to describe the anizotropical materials. The most general way is the fact that a criterion is a function of the components of the stresses tensor in a given base. The chosen form must be intrinsic.

The most general solution generalizes the von Mises criterion, using instead of J(σ) the expression:

$$J_B(\underset{\sim}{\sigma}) = \sqrt{\underset{\sim}{\sigma} : \underset{\approx}{B} : \underset{\sim}{\sigma}} \tag{17}$$

which introduces the tensor of 4th order $\underset{\approx}{B}$. Choosing for $\underset{\approx}{B}$ the tensor J so that $\underset{\approx}{s} = \underset{\approx}{J} : \underset{\approx}{\sigma}$ ($\underset{\approx}{s}$ - the associate deviator for $\underset{\approx}{\sigma}$) is obtain von Mises criterion.

Through considerations of symmetry, as for the elasticity case, the number of free components of the $\underset{\approx}{B}$ tensor could be reduced. Moreover, from the usual conditions, the assurance of the plastic incompressibility must be taken into account. If the material has 3 symmetric perpendicular plans, the terms are zero and there will only remain 6 components.

5. Conclusions

The general equations which describe one of the materials behaviour show the nature of the viscoelasticity, plasticity and viscoplasticity models, the last two having in common the existence of an elasticity field.

It must be mentioned the fact that the deformation or the plastic flow is momentary, while the flow is being delayed. This thing has important consequences in writing the elastic – viscous – plastic behaviour. The effects should not be neglected, because they are will determined.

The majority of these effects (the oldening, the interactions with the environment, etc.) is well established and represents the object of simulations, specific to each studied case. The criterias used for describing the behaviour, as well as the flow laws, must be chosen according to the studied material, its type, the presence of irregularities, fissures, discontinuities, structural defects and what is important especially for rocks, is their anisotropy. A nowadays case in geotechnic is the one of the izotropical materials, which's criterion must be written according to the main normal stresses, which are normal stresses and tangential on a perpendicular face on the axe of schistosity, meaning a parallel face with the izotropical plan of schistosity.

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