ГОДИШНИК на Минно-геоложкия университет "Св. Иван Рилски", Том 48, Св. III, Механизация, електрификация и автоматизация на мините, 2005
ANNUAL of the University of Mining and Geology "St. Ivan Rilski", Vol. 48, Part III, Mechanization, electrification and automation in mines, 2005

DIRECT AND INVERSE PROBLEMS FOR THE LINEAR CONVECTIVE DIFFUSION EQUATION GOVERNING MIGRATION OF GROUND WATER POLLUTION

Ivan Dimov, Petko Lalov

University of Mining and Geology "St. Ivan Rilski", 1700 Sofia, Bulgaria, ***vessad@mgu.bg****;* ***petko@mgu.bg***

ABSTRACT. In this paper one dimensional migration of ground water pollution has been considered. For the linear convective diffusion equation governing of this phenomenon are posed and solved in the domain {0≤x<∞ ; 0≤t≤T }the next problems:

1. *Direct problem*. At the initial moment, when the filtration flow is not polluted, a pollutant source starts to act c(0,t)=f(t) at x=0. We seek for the pollution distribution downstream.
2. *Inverse problem* For the measured pollution at point x=L, i.e. c(L,t), let us determine the characteristics of the pollution source c(0,t)=f(t).
3. *Inverse coefficient problem* For measured c(L,t) and given pollution source characteristics c(0,t)=f(t), let us determine the filtration rate v and convective diffusion coefficient D.

ПРАВИ И ОБРАТНИ ЗАДАЧИ ЗА ЛИНЕЙНОТО УРАВНЕНИЕ НА КОНВЕКТИВНАТА ДИФУЗИЯ, КОЕТО ОПИСВА МИГРАЦИЯТА НА ЗАМЪРСИТЕЛИТЕ НА ПОДПОЧВЕНИТЕ ВОДИ

Иван Димов, Петко Лалов

Минно-геоложки университет "Св. Иван Рилски", 1700 София, ***vessad@mgu.bg****;* ***petko@mgu.bg***

РЕЗЮМЕ. В статията е разгледана едномерната миграция на замърсителите на подпочвените води. За линейното конвективно-дифузионно уравнение, описващо разглежданото явление, са поставени и решени в областта {0≤x<∞ ; 0≤t≤T }, следните задачи:

1. *Права задача*. В началния момент, когато течението не е замърсено, източникът на замърсяването започва да действа c(0,t)=f(t) в точката x=0. Ние търсим разпределението на замърсителя при t>0;
2. *Обратна задача*. По измерения замърсител в точката x=L, т.е. c(L,t), да се определят характеристиките на източника на замърсяване c(0,t)=f(t);
3. *Обратна коефициентна задача*. По измерена концентрация c(L,t) и дадени характеристики на източника на замърсяване c(0,t)=f(t), да се определи скоростта на филтрацията и коефициента на конвективната дифузия.

Introduction

The linear convective diffusion equation, describing the migration of ground water pollution has the form:

$$\frac{\partial c(x,t)}{\partial t} + v\frac{\partial c(x,t)}{\partial x} - D\frac{\partial^2 c(x,t)}{\partial x^2} = 0 \qquad (1)$$

where c[m/L^3] is the pollutant concentration (chemical, biological, radioactive), v[L/T] – filtration rate, D[L^2/T]- convective diffusion coefficient, t[T]- time, x[L]- spatial variable.

In solving concrete problems for equation (1) we should add initial, boundary or mixed conditions. Analyzing the effect of these conditions on the solution to the given problem, J.Hadamard reached the conclusion that if the problem has a physical sense, then its solution should exist and be unique and stable, i.e. the slight deviations in the initial-boundary condition should lead to slight changes in the solution. Problems whose solutions satisfy the above-mentioned conditions we call *well-posed* in Hadamard's sense. If even one of the conditions is not satisfied, then the problem is *ill-posed* (*incorrect*).

Let us introduce two more basic concepts: *direct problem* - for known causes we determine the effect; inverse problem - for a given effect we determine the causes.

The inverse problems are usually *ill-posed*. We cannot achieve stable solutions by classical mathematical means. We need regularization by which we can "return" the problem to the class of well-posed ones. We achieve that by A.H.Tikhonov's regularization method [3].

Here we treat the following direct and inverse problems for a one- imensional filtration flow in the domain {0 ≤x≤ ∞, 0 ≤ t≤ T}:

Problem A (Direct problem). At the initial moment, when the filtration flow is not polluted, a pollutant source starts to act c(0,t)=f(t) at x=0. We seek for the pollutant distribution downstream.

Problem B (Inverse problem). For measured pollution at point x=L, i.e. c(L,t), let us determine the characteristics of the pollution source c(0,t)=f(t).

Problem C (Inverse coefficient problem). For measured c(L,t) and given pollution source characteristics c(0,t)=f(t), let us determine the filtration rate v and convective diffusion coefficient D.

Mathematical background and computer simulation

Problem A. In this case we seek for the solution to equation (1) under the following conditions:

$$c(x,0)=0 \qquad 0\leq x<\infty \tag{2}$$

$$c(0,t)=f(t) \qquad 0\leq t\leq T \tag{3}$$

To achieve our aim, we first substitute the variables

$$\xi = \frac{v}{D}x \ ; \tau = \frac{v^2}{D}t$$

$$c(\xi,\tau) = u(\xi,\tau)\exp(-(\tau/4-\xi/2)) \qquad 4)$$

and the problem (1)-(3) becomes[2]

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial \xi^2} \tag{5}$$

$$u(\xi,0)=0 \tag{6}$$

$$u(0,\tau)=f(\tau)\exp(\tau/4)=\psi(\tau) \tag{7}$$

Applying the Laplace's transformation the solution of the (5)-(7) problem is written in the following form:

$$u(\xi,\tau) = \frac{\xi}{2\sqrt{\pi}}\int_0^\tau \frac{\psi(s)}{(\tau-s)^{3/2}}\exp(-\frac{\xi^2}{4(\tau-s)})ds \tag{8}$$

Returning to the variables (x,t) finally we obtain:

$$c(x,t) = \int_0^{\frac{v^2t}{D}} K(s,\frac{v^2t}{D})f(s)ds \tag{9}$$

where

$$K(s,t) = \frac{l_0(x)}{2\sqrt{\pi}}\frac{\exp(-(\frac{l_0^2}{4(t-s)}+\frac{t-s}{4}-\frac{l_0}{2}))}{(t-s)^{3/2}} \ ,$$

$$l_0(x) = \frac{vx}{D} \tag{10}$$

The solution to this problem has an auxiliary character and will serve as a basis for comparison when solving the following problems.

Problem B. In this case we solve equation (1) for known (measured) pollutant concentration values at point x=L, i.e. for a known c(L,t), t>0, we seek for the pollution source c(0,t)=f(t), acting at point x=0. Thus we come to the need for seeking a solution to Volterra's equation (5), which is an ill-posed problem in sense of Hadamard [3].

We achieve the aim set by the following way. We measure the function c(L,t) for a number of values of t by using (9) we obtain the system of equation[2].

$$\int_0^{\frac{v^2t_1}{D}} K(s,\frac{v^2t_1}{D})f(s)ds = c(L,t_1)$$

$$\int_0^{\frac{v^2t_1}{D}} K(s,\frac{v^2t_2}{D})f(s)ds + \int_{\frac{v^2t_1}{D}}^{\frac{v^2t_2}{D}} K(s,\frac{v^2t_2}{D})f(s)ds$$

$$= c(L,t_2)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \tag{11}$$

$$\int_0^{\frac{vt_1^2}{D}} K(s,\frac{v^2t_m}{D})f(s)ds + \int_{\frac{v^2t_1}{D}}^{\frac{vt_2^2}{D}} K(s,\frac{v^2t_m}{D})f(s)ds +$$

$$\ldots\ldots + \int_{\frac{v^2t_{m-1}}{D}}^{\frac{v^2t_m}{D}} K(s,\frac{v^2t_m}{D})f(s)ds = c(L,t_m)$$

with reference to the unknowns $f(t_i)$, i=1,2,…..,m

After suitable discretization from (11) we obtain the system of linear algebraic equation, which will write as

$$Af=c, \tag{12}$$

Where f is the unknown vector; c is the measured vector and A- the system's matrix. This system has the following particular characteristics: a) the vector c is measured with an error δ, b) when solving (12) many operations are performed with the matrix, roundings at cetera, which introduces into the matrix some error h, c) depending on that the system (12) could be defined, ill-defined or over-defined.

Based on the above conditions the system (12) may have a solution, may not have a solution or may have many solutions.

Before we discuss the choice of method for solving (12) let us introduce spherical norm, which applied to the object under consideration has the form [2]

$$\|c\| = \sqrt{\sum_{i=1}^{m} c_i^2} \quad , \quad \|f\| = \sqrt{\sum_{i=1}^{n} f_i^2} \ ,$$

$$\|A\| = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2} \qquad (13)$$

where a_{ij} are the matrix A elements.

We can say that the system (12) may not have a classical solution however generalized solution(pseudo solution), the sense of Legandere-Gauss least square method(LSM). This is found as a result of the solution of the problem(principal of Legandere-Gauss)[3]:

$$\min\|Af - c\| \qquad (14)$$

It found the system (12) has many solutions. It is possible however to find one solution which is unique, when besides (14) apply the condition

$$\min\|f\| \qquad (15)$$

which means that the we require that the unknown vector f is with minimal norm. This solution we call normal generalized solution, however although it is unique the solution is not stable[3]. That means that looking for a solution of system (12) by satisfying the conditions (14)-(15) is an ill-posed problem.

A stable normal pseudo-solution of (12) could be obtained bu the Tikhonov's method of regularization by founding that element f^{α}, which satisfy the condition,

$$\inf M^{\alpha}[f] = \inf\left\{\|A_h f - c_\delta\|^2 + \alpha\|f\|^2\right\} \qquad (16)$$

where α is a parameter of regularization, defied as the root of the equation

$$\|A_h f^{\alpha} - c_\delta\| = \mu_{h\delta} + \delta + h\|f^{\alpha}\|$$

$$\mu_{h\delta} = \inf\left\{\|A_h f - c_\delta\| + \delta + h\|f\|\right\}, \qquad (17)$$

$M^{\alpha}[f]$ is the Tikhonov's functional(which is concave in downward direction[1]), A_h- is a the matrix which elements have been measured with an error h, which means that $\|A_h - A\| \le h, c_\delta$ – is the vector (right part) measured with an error δ, thus $\|c_\delta - c\| \le \delta$.

Hence the solution we seek has the form:

$$f^{\alpha} = (A_h^T A_h + \alpha E)^{-1} A_h^T c_\delta \qquad (18)$$

where A_h^T is the transposed matrix of the matrix A_h, E-unit matrix, α parameter defined according to condition (17) (for details see [3]).

We implement the regularization method by a special computer programe REGMOD. We would like to mention in addition that: a) without defining the measurement errors we can not solve ill-posed problems [4]; b) in order to solve the problem defined here the classic mathematical method (LSM, Singular Value Decomposition (SVD), Hansen's method etc.) could not be applied.

Problem C. In this case, for pollutant concentration c(L,t) measured at point x=L, and a known source c(0,t)=f(t), we determine simultaneously the filtration rate v and the turbulent diffusion coefficient D.

We achieve the aim by minimizing the function

$$F(v,D) = \sum_{j=1}^{m}\left[\int_0^{\frac{v^2 t}{D}} K\left(s,\frac{v^2 t_j}{D}\right) f(s)\,ds - c(L,t_j)\right]^2 \qquad (19)$$

For finding this minimum in the program realization we used Hooke-Jeeves's method.

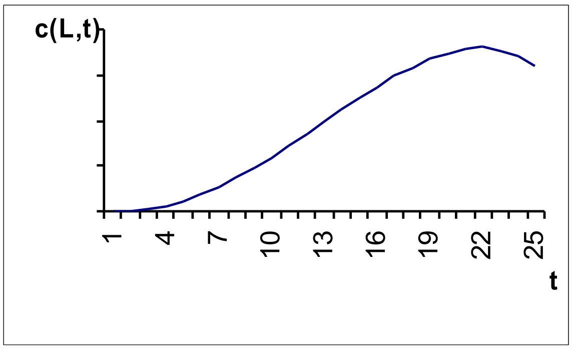
Numerical experiments

We assume the following synthetic data:
L=100m, T=200 day, D=25m^2/day , v=1m/day.

$$f(t) = \begin{cases} t/100 \quad , & t < 100\,day \\ 1 \quad , & 100\,day \le t \le 150\,day \\ (200-t)/50, & 150\,day \le t \le 200\,day \end{cases}$$

Fig. 1 shows the graph of the direct problem solution (problem A) at the point x=L.

Fig.1.



When we disturb the function c(L,t) and solve the inverse (incorrect) problem, i.e. when we solve the system of linear algebraic equations obtained, which we described in treating problem B by LSM and Tikhonov's regularization emthod (in case of errors in the matrix elements and the right-hand part $h=10^{-5}$ and $\delta=10^{-2}$), we obtain the results shown in Fig. 2.

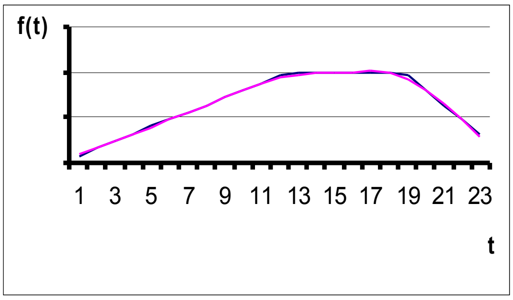
It can be noted that because of the incorrectness of the problem LSM is inapplicable here(Fig.2-a seesaw line). None of the classical methods is applicable.

The problem C is solved for the same data, taking for initial values of v and D

$v=1m/day$, $D=20m^2/day$

After minimizing (7), we obtain:

$v=2.0002$ m/day , $D=25.518$ m^2/day



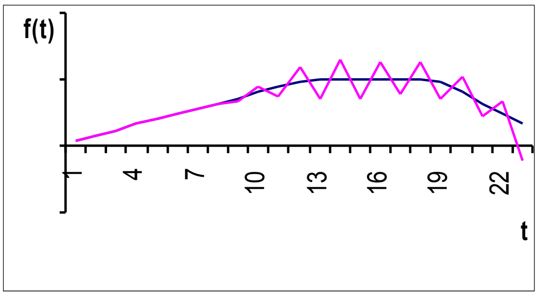


Fig 2-a. exact and regularized solution
Fig 2-b. exact and LSM's solution

Conclusion

Within the framework of the one-dimensional linear convective diffusion model, describing the migration of ground water pollution, we have solved two basic problems which offer the following possibilities:

- for measured pollution concentration at a give point in a number of moments, to determined the characteristics of the pollution source;
- for measured concentration in a number of moments at a given point and a known pollution source, to determine simultaneously both the filtration rate and the convective diffusion coefficient.

We can conclude that within the framework of the adopted convective diffusion model, the results we have obtained are a definite progress in the migration theory of ground water pollution.

References

1. Il'in V.A, E.G. Pozniak. Linear algebra, Nauka, Moscow, 1978, pp. 302 (in Russian).
2. Lalov P. The determination of the characteristics to gas source and coefficient of the Convective Diffusion Equation as an incorrect Problem, Proceedings of the Seventh International Mine Ventilation Congress, June 17-22, 2001 Cracow, Poland.
3. Tikhonov, A.N and V.A. Arsenin. Solution of ill-posed Problems, V.H.Winston & Sons. Washington D.C., 1977, pp 255.
4. Leonov A.S., A.G. Yagola. Can an ill-posed problem be solved if the data errors is unknown?, Moscow University, Physics Bulletin, vol.50, №4, 1995, pp.25-28.

Recommended for publication by Department of Informatics, Faculty of Mining Electromechanics