

## ВЪРХУ ЕКВИВАЛЕНТНОСТТА НА СИСТЕМИ ДИФЕРЕНЦИАЛНИ УРАВНЕНИЯ ВЪЗНИКВАЩИ В ЕЛЕКТРОМАГНИТНАТА ЗАДАЧА ЗА ДВЕ ТЕЛА

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### РЕЗЮМЕ

Разглеждат се два типа системи от уравнения на движението, възникващи в електромагнитната задача за две тела (Synge, 1940; Synge, 1960) и е показана тяхната еквивалентност.

В представената статия се разглеждат два типа системи от уравнения на движението, възникващи в електромагнитната задача за две тела (Synge, 1940; Synge, 1960) и формулирани в (Angelov, 2002).

Най-напред да припомним някои означения и резултати (Angelov, 2002; Angelov, 2000).

### Уравненията на движение въведени от J. L. Synge

Означаваме както в (Synge, 1940) с

$$x^{(p)} = (x_1^{(p)}(t), x_2^{(p)}(t), x_3^{(p)}(t), x_4^{(p)}(t) = ict)$$

$$(p = 1, 2, i^2 = -1)$$

пространствено-временните координати на движещите се частици, с  $m_p$  - техните собствени маси, с  $e_p$  - техните заряди,  $c$  - скоростта на светлината. Координатите на векторите на скоростта са

$$u^{(p)} = (u_1^{(p)}(t), u_2^{(p)}(t), u_3^{(p)}(t)) \quad (p = 1, 2).$$

Координатите на единичните допирателни вектори са

$$\lambda_\alpha^{(p)} = \frac{\gamma_p u_\alpha^{(p)}(t)}{c} = \frac{u_\alpha^{(p)}(t)}{\Delta_p} \quad (\alpha = 1, 2, 3), \lambda_4^{(p)} = i\gamma_p = \frac{ic}{\Delta_p},$$

където

$$\gamma_p = \left( 1 - \frac{1}{c^2} \sum_{\alpha=1}^3 [u_\alpha^{(p)}(t)]^2 \right)^{-\frac{1}{2}}, \Delta_p = \left( c^2 - \sum_{\alpha=1}^3 [u_\alpha^{(p)}(t)]^2 \right)^{\frac{1}{2}}.$$

Следователно  $\gamma_p = c/\Delta_p$ .

Означаваме скаларното произведение в пространство-то на Минковски с  $\langle \dots \rangle_4$ , а с  $\langle \dots \rangle$  - скаларното произведение в 3-мерното евклидово подпространство.

Уравненията на Synge са:

$$m_p \frac{d\lambda_r^{(p)}}{ds_p} = \frac{e_p}{c^2} F_{rn}^{(p)} \lambda_n^{(p)} \quad (r = 1, 2, 3, 4) \quad (1)$$

където  $ds_p = \frac{c}{\gamma_p} dt = \Delta_p dt$  ( $p = 1, 2$ ). Да отбележим, че в

(1) се сумира по  $n$  ( $n = 1, 2, 3$ ). Елементите на електромагнитния тензор -  $F_{rn}^{(p)}$ , се получават от закъсняващите потенциали (на Lienard-Wiechert)

$$A_r^{(p)} = - \frac{e_p \lambda_r^{(p)}}{\langle \lambda^{(p)}, \xi^{(pq)} \rangle_4} \quad (r = 1, 2, 3, 4), \text{ т.е.}$$

$$F_{rn}^{(p)} = \frac{\partial A_n^{(p)}}{\partial x_r^{(p)}} - \frac{\partial A_r^{(p)}}{\partial x_n^{(p)}} \quad \xi^{(pq)}.$$

Означаваме с  $\xi^{(pq)}$  изотропния вектор (вж. [1], [2]):

$$\xi^{(pq)} = (x_1^{(p)}(t) - x_1^{(q)}(t - \tau_{pq}(t)),$$

$$x_2^{(p)}(t) - x_2^{(q)}(t - \tau_{pq}(t)), x_3^{(p)}(t) - x_3^{(q)}(t - \tau_{pq}(t)), ict_{pq}(t))$$

където  $\langle \xi^{(p,q)}, \xi^{(p,q)} \rangle_4 = 0$ , или

$$\tau_{pq}(t) = \frac{1}{c} \left( \sum_{\beta=1}^3 [x_{\beta}^{(p)}(t) - x_{\beta}^{(q)}(t - \tau_{pq}(t))]^2 \right)^{\frac{1}{2}} \quad (*)$$

((pq) = (12), (21)).

Пресмятайки  $F_{rn}^{(p)}$  както в (Angelov, 1990) можем да напишем уравненията от (1) във вида

$$\begin{aligned} \frac{d\lambda_{\alpha}^{(p)}}{ds_p} = \frac{Q_p}{c^2} & \left[ \frac{\xi_{\alpha}^{(pq)} \langle \lambda^{(p)}, \lambda^{(q)} \rangle_4 - \lambda_{\alpha}^{(q)} \langle \lambda^{(p)}, \xi^{(pq)} \rangle_4}{\langle \lambda^{(q)}, \xi^{(pq)} \rangle_4^3} \right] 1 + \\ & + \left\langle \xi^{(pq)}, \frac{d\lambda^{(q)}}{ds_q} \right\rangle_4 + \frac{1}{\langle \lambda^{(q)}, \xi^{(pq)} \rangle_4^2} \left[ \langle \lambda^{(p)}, \xi^{(pq)} \rangle_4 \frac{d\lambda^{(q)}}{ds_q} \right. \\ & \left. - \left\langle \xi^{(pq)}, \frac{d\lambda^{(q)}}{ds_q} \right\rangle_4 \xi_{\alpha}^{(pq)} \right] \quad (\alpha = 1,2,3) \quad (2.\alpha) \end{aligned}$$

$$\begin{aligned} \frac{d\lambda_4^{(p)}}{ds_p} = \frac{Q_p}{c^2} & \left[ \frac{\xi_4^{(pq)} \langle \lambda^{(p)}, \lambda^{(q)} \rangle_4 - \lambda_4^{(q)} \langle \lambda^{(p)}, \xi^{(pq)} \rangle_4}{\langle \lambda^{(q)}, \xi^{(pq)} \rangle_4^3} \right] 1 + \\ & + \left\langle \xi^{(pq)}, \frac{d\lambda^{(q)}}{ds_q} \right\rangle_4 + \frac{1}{\langle \lambda^{(q)}, \xi^{(pq)} \rangle_4^2} \left[ \langle \lambda^{(p)}, \xi^{(pq)} \rangle_4 \frac{d\lambda^{(q)}}{ds_q} \right. \\ & \left. - \left\langle \xi^{(pq)}, \frac{d\lambda^{(q)}}{ds_q} \right\rangle_4 \xi_4^{(pq)} \right] \quad (2.4) \end{aligned}$$

където  $Q_p = \frac{e_1 \cdot e_2}{m_p}$ , (p = 1,2).

Означаваме  $u^{(q)} \equiv u^{(q)}(t - \tau_{pq})$ ,

$$\begin{aligned} \lambda^{(q)} &= (\gamma_{pq} u_1^{(q)} / c, \gamma_{pq} u_2^{(q)} / c, \gamma_{pq} u_3^{(q)} / c, i\gamma_{pq}) = \\ &= (u_1^{(q)} / \Delta_{pq}, u_2^{(q)} / \Delta_{pq}, u_3^{(q)} / \Delta_{pq}, ic / \Delta_{pq}), \end{aligned}$$

където

$$\gamma_{pq} = \left( 1 - \frac{1}{c^2} \sum_{\alpha=1}^3 [u_{\alpha}^{(q)}(t - \tau_{pq}(t))]^2 \right)^{-\frac{1}{2}},$$

$$\Delta_{pq} = \left( c^2 - \sum_{\alpha=1}^3 [u_{\alpha}^{(q)}(t - \tau_{pq}(t))]^2 \right)^{\frac{1}{2}} \text{ и}$$

$$\begin{aligned} \frac{d\lambda_{\alpha}^{(p)}}{ds_p} &= \frac{d\left( \frac{\gamma_p}{c} u_{\alpha}^{(p)} \right)}{\frac{c}{\gamma_p} dt} = \frac{d\left( \frac{u_{\alpha}^{(p)}}{\Delta_p dt} \right)}{\Delta_p dt} = \\ &= \frac{1}{\Delta_p^2} \dot{u}_{\alpha}^{(p)} + \frac{u_{\alpha}^{(p)}}{\Delta_p^4} \langle u^{(p)}, \dot{u}^{(p)} \rangle \quad (\alpha = 1,2,3) \end{aligned}$$

$$\frac{d\lambda_4^{(p)}}{ds_p} = \frac{d(i\gamma_p)}{\frac{c}{\gamma_p} dt} = \frac{icd\left( \frac{1}{\Delta_p} \right)}{\Delta_p dt} = \frac{ic}{\Delta_p^4} \langle u^{(p)}, \dot{u}^{(p)} \rangle,$$

където точката означава диференциране по t.

Като в (Angelov, 2000), където се доказва, че 4-то и 8-то уравнения са следствие от останалите, получаваме система от 6 уравнения. Сега можем да формулираме задача с начални условия по следния начин: да се намерят неизвестните скорости  $u_{\alpha}^{(p)}(t)$  (p = 1,2;  $\alpha = 1,2,3$ ), които удовлетворяват уравненията на движение  $(3_{1\alpha}), (3_{2\alpha})$ , както следва:

$$\begin{aligned} \frac{1}{\Delta_1} \dot{u}_{\alpha}^{(1)} + \frac{u_{\alpha}^{(1)}}{\Delta_1^3} \langle u^{(1)}, \dot{u}^{(1)} \rangle &= \\ &= \frac{Q_1}{c^2} \left[ \frac{[c^2 - \langle u^{(1)}, u^{(2)} \rangle] \xi_{\alpha}^{(12)} - [c^2 \tau_{12} - \langle u^{(1)}, \xi^{(12)} \rangle] u_{\alpha}^{(2)}}{[c^2 \tau_{12} - \langle u^{(1)}, \xi^{(12)} \rangle]^3} \right] \times \\ &\times \frac{\Delta_{12}^4 + D_{12} \Delta_{12}^2 \langle \xi^{(12)}, \dot{u}^{(2)} \rangle + \left( \langle \xi^{(12)}, u^{(2)} \rangle - c^2 \tau_{12} \right) \langle u^{(2)}, \dot{u}^{(2)} \rangle}{\Delta_{12}^2} + \\ &+ D_{12} \frac{\left( \langle u^{(1)}, \xi^{(12)} \rangle - c^2 \tau_{12} \right) \dot{u}_{\alpha}^{(2)} - \langle u^{(1)}, \dot{u}^{(2)} \rangle \xi_{\alpha}^{(12)}}{\left[ c^2 \tau_{12} - \langle u^{(2)}, \xi^{(12)} \rangle \right]^2} + \quad (3_{1\alpha}) \\ &+ \frac{D_{12}}{\Delta_{12}^2} \frac{\left( \langle u^{(1)}, \xi^{(12)} \rangle - c^2 \tau_{12} \right) u_{\alpha}^{(2)} \langle u^{(2)}, \dot{u}^{(2)} \rangle}{\left[ c^2 \tau_{12} - \langle u^{(2)}, \xi^{(12)} \rangle \right]^2} + \\ &+ \frac{D_{12}}{\Delta_{12}^2} \frac{\left( c^2 - \langle u^{(1)}, u^{(2)} \rangle \right) \xi_{\alpha}^{(12)} \langle u^{(2)}, \dot{u}^{(2)} \rangle}{\left[ c^2 \tau_{12} - \langle u^{(2)}, \xi^{(12)} \rangle \right]^2} \end{aligned}$$

$$\begin{aligned}
& \frac{1}{\Delta_2} \dot{u}_\alpha^{(2)} + \frac{u_\alpha^{(2)}}{\Delta_1^3} \langle u^{(2)}, \dot{u}^{(2)} \rangle = \\
& = \frac{Q_1}{c^2} \left\{ \frac{[c^2 - \langle u^{(1)}, u^{(2)} \rangle] \xi_\alpha^{(21)} - [c^2 \tau_{12} - \langle u^{(1)}, \xi^{(21)} \rangle] u_\alpha^{(1)}}{[c^2 \tau_{12} - \langle u^{(1)}, \xi^{(12)} \rangle]^3} \times \right. \\
& \times \frac{\Delta_{21}^4 + D_{21} \Delta_{21}^2 \langle \xi^{(21)}, \dot{u}^{(1)} \rangle + \left( \langle \xi^{(21)}, u^{(1)} \rangle - c^2 \tau_{21} \right) \langle u^{(1)}, \dot{u}^{(1)} \rangle}{\Delta_{21}^2} + \\
& + D_{21} \frac{\left( \langle u^{(2)}, \xi^{(21)} \rangle - c^2 \tau_{21} \right) \dot{u}_\alpha^{(1)} - \langle u^{(2)}, \dot{u}^{(1)} \rangle \xi_\alpha^{(21)}}{[c^2 \tau_{21} - \langle u^{(1)}, \xi^{(21)} \rangle]^2} + \\
& + \frac{D_{21}}{\Delta_{21}^2} \cdot \frac{\left( \langle u^{(2)}, \xi^{(21)} \rangle - c^2 \tau_{21} \right) u_\alpha^{(1)} \langle u^{(1)}, \dot{u}^{(1)} \rangle}{[c^2 \tau_{21} - \langle u^{(1)}, \xi^{(21)} \rangle]^2} + \quad (3_{2\alpha}) \\
& \left. + \frac{D_{21}}{\Delta_{21}^2} \cdot \frac{\left( c^2 - \langle u^{(2)}, u^{(1)} \rangle \right) \xi_\alpha^{(21)} \langle u^{(1)}, \dot{u}^{(1)} \rangle}{[c^2 \tau_{21} - \langle u^{(1)}, \xi^{(21)} \rangle]^2} \right\}
\end{aligned}$$

Припомняме, че в горните уравнения (3<sub>1α</sub>)

$$u^{(1)} = u^{(1)}(t), u^{(2)} = u^{(2)}(t - \tau_{12}),$$

докато в (3<sub>2α</sub>)  $u^{(2)} = u^{(2)}(t)$ ,  $u^{(1)} = u^{(1)}(t - \tau_{21})$ , а закъсняващата функция  $\tau_{pq}(t)$  удовлетворява функционалните уравнения (\*) за  $t \in (-\infty, \infty)$ .

За  $t \leq 0$  функциите  $u_\alpha^{(p)}(t)$  са дадени:

$$u_\alpha^{(p)}(t) = \bar{u}_\alpha^{(p)}(t), t \leq 0,$$

където

$$\bar{u}_\alpha^{(p)}(t) = \frac{d\bar{x}_\alpha^{(p)}(t)}{dt}, t \leq 0.$$

Това означава, че за дадените траектории  $(\bar{x}_1^{(1)}(t), \bar{x}_2^{(1)}(t), \bar{x}_3^{(1)}(t)), (\bar{x}_1^{(2)}(t), \bar{x}_2^{(2)}(t), \bar{x}_3^{(2)}(t))$  за  $t \leq 0$  трябва да се намерят траектории удовлетворяващи горната система от уравнения при

$t > 0$ . (Припомняме, че  $x_\alpha^{(p)}(t) = x_{\alpha 0}^{(p)} + \int_0^t u_\alpha^{(p)}(s) ds$ ,

където  $x_{\alpha 0}^{(p)}$  са началните координати).

### Задача на Кеплер в полярни координати

Сега ще разгледаме равнинно движение в координатната система  $Ox_2x_3$  за горните уравнения.

Предполагаме, че първата частица  $P_1$  е фиксирана в началото  $O(0,0,0)$ , т.е.

$$P_1 : \begin{cases} x_1^{(1)}(t) = 0 \\ x_2^{(1)}(t) = 0, \quad t \in (-\infty, \infty). \\ x_3^{(1)}(t) = 0 \end{cases} \quad \text{Следователно, по}$$

$$\text{необходимост, } \begin{cases} \bar{x}_1^{(1)}(t) = 0 \\ \bar{x}_2^{(1)}(t) = 0. \\ \bar{x}_3^{(1)}(t) = 0 \end{cases} \quad \text{Премаивайки в полярни}$$

$$\text{координати полагаме } P_1 : \begin{cases} x_1^{(2)}(t) = 0 \\ x_2^{(2)}(t) = \rho(t) \cos \varphi(t) \\ x_3^{(2)}(t) = \rho(t) \sin \varphi(t) \end{cases}, \quad \text{където}$$

$$\rho(t) > 0.$$

След преобразувания направени в (Angelov, 2002) получаваме следната система от втори ред:

$$\ddot{\rho}(t) = \rho(t) \dot{\varphi}^2(t) + \frac{Q_2}{c^3} \cdot \frac{[c^2 - \dot{\rho}^2(t)] \sqrt{c^2 - \dot{\rho}^2(t) - \rho^2(t) \dot{\varphi}^2(t)}}{\rho^2(t)} \quad (4)$$

$$\ddot{\varphi}(t) = -\frac{2\dot{\rho}(t)\dot{\varphi}(t)}{\rho(t)} \cdot \left[ 1 + \frac{Q_2 \sqrt{c^2 - \dot{\rho}^2(t) - \rho^2(t) \dot{\varphi}^2(t)}}{2c^3 \rho(t)} \right]$$

при  $t > 0$  и начални условия

$$\rho(0) = \rho_0, \dot{\rho}(0) = \dot{\rho}_0, \varphi(0) = \varphi_0, \dot{\varphi}(0) = \dot{\varphi}_0.$$

От друга страна, започвайки от началната форма на уравненията на Synge (Angelov, 2000), получаваме за кеплеровата задача следните уравнения на движение:

$$\frac{d(\gamma_2 u_\alpha^{(2)})}{dt} = \frac{Q_2 \xi_\alpha^{(21)}}{\rho^3} \quad (\alpha = 1,2,3) \quad (5_\alpha)$$

Но  $\xi^{(21)} = (0, \rho(t) \cos \varphi(t), \rho(t) \sin \varphi(t))$ . Тогава с интегриране от 0 до  $t$  получаваме

$$\gamma_2(t) u_\alpha^{(2)}(t) - \gamma_2^0 u_\alpha^{(2)}(0) = Q_2 \int_0^t \frac{\xi_\alpha^{(21)}(s)}{\rho^3(s)} ds \quad \text{или}$$

$$\begin{cases} \dot{\rho}(t) \cos \varphi(t) - \rho(t) \dot{\varphi}(t) \sin \varphi(t) = \frac{1}{\gamma_2(t)} \left[ Q_2 \int_0^t \frac{\cos \varphi(s)}{\rho^2(s)} ds + C_2 \right] \\ \dot{\rho}(t) \sin \varphi(t) + \rho(t) \dot{\varphi}(t) \cos \varphi(t) = \frac{1}{\gamma_2(t)} \left[ Q_2 \int_0^t \frac{\sin \varphi(s)}{\rho^2(s)} ds + C_3 \right] \end{cases} \quad (6)$$

$$\text{където } C_2 = \gamma_2^0 (\dot{\rho}_0 \cos \varphi_0 - \rho_0 \dot{\varphi}_0 \sin \varphi_0),$$

$$C_3 = \gamma_2^0 (\dot{\rho}_0 \sin \varphi_0 + \rho_0 \dot{\varphi}_0 \cos \varphi_0),$$

$$\gamma_2^0 = \frac{1}{\sqrt{1 - \frac{1}{c^2} (\dot{\rho}_0^2 + \rho_0^2 \dot{\varphi}_0^2)}}.$$

Системите (6) и (5<sub>α</sub>) са еквивалентни, тъй като дясната страна на (5<sub>α</sub>) е непрекъсната вектор-функция на  $t$  (което е необходимо и достатъчно условие за еквивалентност на двете системи). Наистина вектор-функцията  $\vec{F}(t) = \left( \frac{\cos \varphi(t)}{\rho^2(t)}, \frac{\sin \varphi(t)}{\rho^2(t)} \right)$  е непрекъсната,

$$\begin{aligned} \text{тъй като } & \left| \vec{F}(t) - \vec{F}(t_0) \right|^2 = \\ & = \frac{1}{\rho^4(t)} + \frac{1}{\rho^4(t_0)} - \frac{2 \cos(\varphi(t) - \varphi(t_0))}{\rho^2(t) \rho^2(t_0)} \xrightarrow{t \rightarrow t_0} 0. \end{aligned}$$

Разбира се  $\frac{d}{dt} (\gamma_2 u_\alpha^{(2)}) = \dot{\gamma}_2 u_\alpha^{(2)} + \gamma_2 \dot{u}_\alpha^{(2)}$ , ( $\alpha = 2, 3$ )

$$\dot{\gamma}_2 = \frac{d}{dt} \left( \frac{c}{\Delta_2} \right) = \frac{c}{\Delta_2^3} \langle u, \dot{u} \rangle.$$

Тогава от (5<sub>α</sub>) (с  $u \equiv u^{(2)}$ ) за краткост) получаваме:

$$\begin{cases} \frac{c}{\Delta_2^3} \langle u, \dot{u} \rangle u_2 + \frac{c}{\Delta_2} \dot{u}_2 = \frac{Q_2 \cos \varphi}{\rho^2} \\ \frac{c}{\Delta_2^3} \langle u, \dot{u} \rangle u_3 + \frac{c}{\Delta_2} \dot{u}_3 = \frac{Q_2 \sin \varphi}{\rho^2}. \end{cases} \quad (7)$$

Умножавайки първото уравнение с  $u_2$ , второто с  $u_3$  и сумирайки, получаваме уравнението

$$\frac{c}{\Delta_2^3} \langle u, \dot{u} \rangle \left( \langle u, u \rangle + \Delta_2^2 \right) = \frac{Q_2}{\rho^2} (u_2 \cos \varphi + u_3 \sin \varphi),$$

$$\text{т.е. } \frac{c}{\Delta_2^3} \langle u, \dot{u} \rangle = \frac{Q_2 \dot{\rho}}{c^2 \rho^2}.$$

$$\text{С други думи получихме връзката } \dot{\gamma}_2 = \frac{Q_2 \dot{\rho}}{c^2 \rho^2}.$$

Умножавайки първото уравнение на (7) с  $\cos \varphi$ , а второто със  $\sin \varphi$  и сумирайки, и след това първото уравнение с  $-\sin \varphi$ , второто с  $\cos \varphi$  и сумирайки, получаваме

$$\text{системата } \begin{cases} \dot{\gamma}_2 \dot{\rho} + \gamma_2 (\ddot{\rho} - \rho \dot{\varphi}^2) = \frac{Q_2}{\rho^2}, \\ \dot{\gamma}_2 \rho \dot{\varphi} + \gamma_2 (2\dot{\rho} \dot{\varphi} + \rho \ddot{\varphi}) = 0 \end{cases}, \text{ или (като заместим}$$

$$\dot{\gamma}_2 = \frac{Q_2 \dot{\rho}}{c^2 \rho^2}:$$

$$\begin{cases} \gamma_2 (\ddot{\rho} - \rho \dot{\varphi}^2) = \frac{Q_2}{c^2 \rho^2} (c^2 - \dot{\rho}^2) \\ \gamma_2 (\rho \ddot{\varphi} + 2\dot{\rho} \dot{\varphi}) = -\frac{Q_2}{c^2 \rho} \dot{\rho} \dot{\varphi} \end{cases} \quad (8)$$

Очевидно системите (8) и (4) са една и съща система, с точност до прехвърляне на събираеми, тъй като

$$\gamma_2 = \frac{c}{\Delta_2} = \frac{c}{\sqrt{c^2 - \dot{\rho}^2 - \rho^2 \dot{\varphi}^2}}.$$

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## ON THE EQUIVALENCE OF DIFFERENTIAL SYSTEMS ARISING IN ELECTROMAGNETIC TWO-BODY PROBLEM

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### SUMMARY

The equivalence of two systems of equations of motion arising in electromagnetic two-body problem is obtained.

In the present note we consider two systems of equations of motion arising in electromagnetic two-body problem (Synge, 1940; Synge, 1960) and formulated in (Angelov, 2002).

First we recall some denotations and results from (Angelov, 2002; Angelov, 2000) concerning

### J. L. Synge's equations of motion

As in (Synge, 1940) we denote by

$$x^{(p)} = (x_1^{(p)}(t), x_2^{(p)}(t), x_3^{(p)}(t), x_4^{(p)}(t) = ict)$$

( $p = 1, 2$ ,  $i^2 = -1$ ) the space-time coordinates of the moving particles, by  $m_p$  - their proper masses, by  $e_p$  - their charges,  $c$  - the speed of the light. The coordinates of the velocity vectors are

$$u^{(p)} = (u_1^{(p)}(t), u_2^{(p)}(t), u_3^{(p)}(t)) \quad (p = 1, 2).$$

The coordinates of the unit tangent vectors to the world-lines are

$$\lambda_\alpha^{(p)} = \frac{\gamma_p u_\alpha^{(p)}(t)}{c} = \frac{u_\alpha^{(p)}(t)}{\Delta_p} \quad (\alpha = 1, 2, 3), \lambda_4^{(p)} = i\gamma_p = \frac{ic}{\Delta_p},$$

where

$$\gamma_p = \left( 1 - \frac{1}{c^2} \sum_{\alpha=1}^3 [u_\alpha^{(p)}(t)]^2 \right)^{-\frac{1}{2}}, \Delta_p = \left( c^2 - \sum_{\alpha=1}^3 [u_\alpha^{(p)}(t)]^2 \right)^{\frac{1}{2}}.$$

It follows  $\gamma_p = c/\Delta_p$ .

By  $\langle \dots \rangle_4$  we denote the scalar product in the Minkowski space, while by  $\langle \dots \rangle$  - the scalar product in 3-dimensional Euclidean subspace. Synge's equations of motion modeling the interaction of two moving charged particles are the following:

$$m_p \frac{d\lambda_r^{(p)}}{ds_p} = \frac{e_p}{c^2} F_{rn}^{(p)} \lambda_n^{(p)} \quad (r = 1, 2, 3, 4) \quad (1)$$

where the elements of proper time are

$$ds_p = \frac{c}{\gamma_p} dt = \Delta_p dt \quad (p = 1, 2).$$

Recall that in (1) there is a summation in  $n$  ( $n = 1, 2, 3$ ).

The elements  $F_{rn}^{(p)}$  of the electromagnetic tensors are derived by the retarded Lienard-Wiechert potentials

$$A_r^{(p)} = - \frac{e_p \lambda_r^{(p)}}{\langle \lambda^{(p)}, \xi^{(pq)} \rangle_4} \quad (r = 1, 2, 3, 4), \text{ that is}$$

$$F_{rn}^{(p)} = \frac{\partial A_n^{(p)}}{\partial x_r^{(p)}} - \frac{\partial A_r^{(p)}}{\partial x_n^{(p)}} \xi^{(pq)}.$$

By  $\xi^{(pq)}$  we denote the isotropic vectors (cf. Synge, 1940; 1960) drawn into the past:

$$\xi^{(pq)} = (x_1^{(p)}(t) - x_1^{(q)}(t - \tau_{pq}(t)), x_2^{(p)}(t) - x_2^{(q)}(t - \tau_{pq}(t)), x_3^{(p)}(t) - x_3^{(q)}(t - \tau_{pq}(t)), ict_{pq}(t))$$

where  $\langle \xi^{(p,q)}, \xi^{(p,q)} \rangle_4 = 0$  or

$$\tau_{pq}(t) = \frac{1}{c} \left( \sum_{\beta=1}^3 [x_\beta^{(p)}(t) - x_\beta^{(q)}(t - \tau_{pq}(t))]^2 \right)^{\frac{1}{2}} \quad (*)$$

(( $pq$ ) = (12), (21)).

Calculating  $F_{rn}^{(p)}$  as in (Angelov, 1990) we write equations from (2) in the form:

$$\frac{d\lambda_{\alpha}^{(p)}}{ds_p} = \frac{Q_p}{c^2} \left[ \frac{\xi_{\alpha}^{(pq)} \langle \lambda^{(p)}, \lambda^{(q)} \rangle_4 - \lambda_{\alpha}^{(q)} \langle \lambda^{(p)}, \xi^{(pq)} \rangle_4}{\langle \lambda^{(q)}, \xi^{(pq)} \rangle_4^3} \right] + \left\langle \xi^{(pq)}, \frac{d\lambda^{(q)}}{ds_q} \right\rangle_4 + \frac{1}{\langle \lambda^{(q)}, \xi^{(pq)} \rangle_4^2} \left[ \langle \lambda^{(p)}, \xi^{(pq)} \rangle_4 \frac{d\lambda^{(q)}}{ds_q} - \left\langle \xi^{(pq)}, \frac{d\lambda^{(q)}}{ds_q} \right\rangle_4 \xi_{\alpha}^{(pq)} \right] \quad (\alpha = 1,2,3) \quad (2.\alpha)$$

$$\frac{d\lambda_4^{(p)}}{ds_p} = \frac{Q_p}{c^2} \left[ \frac{\xi_4^{(pq)} \langle \lambda^{(p)}, \lambda^{(q)} \rangle_4 - \lambda_4^{(q)} \langle \lambda^{(p)}, \xi^{(pq)} \rangle_4}{\langle \lambda^{(q)}, \xi^{(pq)} \rangle_4^3} \right] + \left\langle \xi^{(pq)}, \frac{d\lambda^{(q)}}{ds_q} \right\rangle_4 + \frac{1}{\langle \lambda^{(q)}, \xi^{(pq)} \rangle_4^2} \left[ \langle \lambda^{(p)}, \xi^{(pq)} \rangle_4 \frac{d\lambda^{(q)}}{ds_q} - \left\langle \xi^{(pq)}, \frac{d\lambda^{(q)}}{ds_q} \right\rangle_4 \xi_4^{(pq)} \right] \quad (2.4)$$

where  $Q_p = \frac{e_1 \cdot e_2}{m_p}$ , ( $p = 1,2$ ). Further on we denote

$$u^{(q)} \equiv u^{(q)}(t - \tau_{pq}),$$

$$\lambda^{(q)} = (\gamma_{pq} u_1^{(q)} / c, \gamma_{pq} u_2^{(q)} / c, \gamma_{pq} u_3^{(q)} / c, i\gamma_{pq}) = (u_1^{(q)} / \Delta_{pq}, u_2^{(q)} / \Delta_{pq}, u_3^{(q)} / \Delta_{pq}, ic / \Delta_{pq}),$$

$$\text{where } \gamma_{pq} = \left( 1 - \frac{1}{c^2} \sum_{\alpha=1}^3 [u_{\alpha}^{(q)}(t - \tau_{pq}(t))]^2 \right)^{-\frac{1}{2}},$$

$$\Delta_{pq} = \left( c^2 - \sum_{\alpha=1}^3 [u_{\alpha}^{(q)}(t - \tau_{pq}(t))]^2 \right)^{\frac{1}{2}} \text{ and}$$

$$\frac{d\lambda_{\alpha}^{(p)}}{ds_p} = \frac{d\left(\frac{\gamma_p}{c} u_{\alpha}^{(p)}\right)}{\frac{c}{\gamma_p} dt} = \frac{d\left(\frac{u_{\alpha}^{(p)}}{\Delta_p dt}\right)}{\Delta_p dt} = \frac{1}{\Delta_p^2} \dot{u}_{\alpha}^{(p)} + \frac{u_{\alpha}^{(p)}}{\Delta_p^4} \langle u^{(p)}, \dot{u}^{(p)} \rangle \quad (\alpha = 1,2,3)$$

$$\frac{d\lambda_4^{(p)}}{ds_p} = \frac{d(i\gamma_p)}{\frac{c}{\gamma_p} dt} = \frac{icd\left(\frac{1}{\Delta_p}\right)}{\Delta_p dt} = \frac{ic}{\Delta_p^4} \langle u^{(p)}, \dot{u}^{(p)} \rangle,$$

where the dot means a differentiation in  $t$ .

Proceeding as in (Angelov, 2000) and proving that 4-th and 8-th equations are a consequence of the rest ones we obtain a system of 6 equations. Now we are able to formulate the initial value problem for the above equations in the following way: to find unknown velocities  $u_{\alpha}^{(p)}(t)$  ( $p = 1,2; \alpha = 1,2,3$ ), for  $t \geq 0$  satisfying equations (3<sub>1α</sub>), (3<sub>2α</sub>) of motion (written in details below):

$$\frac{1}{\Delta_1} \dot{u}_{\alpha}^{(1)} + \frac{u_{\alpha}^{(1)}}{\Delta_1^3} \langle u^{(1)}, \dot{u}^{(1)} \rangle = \frac{Q_1}{c^2} \left[ \frac{c^2 - \langle u^{(1)}, u^{(2)} \rangle \xi_{\alpha}^{(12)} - [c^2 \tau_{12} - \langle u^{(1)}, \xi^{(12)} \rangle] u_{\alpha}^{(2)}}{[c^2 \tau_{12} - \langle u^{(1)}, \xi^{(12)} \rangle]^3} \times \Delta_{12}^4 + D_{12} \Delta_{12}^2 \langle \xi^{(12)}, \dot{u}^{(2)} \rangle + \left( \langle \xi^{(12)}, u^{(2)} \rangle - c^2 \tau_{12} \right) \langle u^{(2)}, \dot{u}^{(2)} \rangle \right] + D_{12} \frac{\left( \langle u^{(1)}, \xi^{(12)} \rangle - c^2 \tau_{12} \right) \dot{u}_{\alpha}^{(2)} - \langle u^{(1)}, \dot{u}^{(2)} \rangle \xi_{\alpha}^{12}}{[c^2 \tau_{12} - \langle u^{(2)}, \xi^{(12)} \rangle]^2} + \quad (3_{1\alpha})$$

$$+ \frac{D_{12}}{\Delta_{12}^2} \cdot \frac{\left( \langle u^{(1)}, \xi^{(12)} \rangle - c^2 \tau_{12} \right) u_{\alpha}^{(2)} \langle u^{(2)}, \dot{u}^{(2)} \rangle}{[c^2 \tau_{12} - \langle u^{(2)}, \xi^{(12)} \rangle]^2} + \frac{D_{12}}{\Delta_{12}^2} \cdot \frac{\left( c^2 - \langle u^{(1)}, u^{(2)} \rangle \right) \xi_{\alpha}^{(12)} \langle u^{(2)}, \dot{u}^{(2)} \rangle}{[c^2 \tau_{12} - \langle u^{(2)}, \xi^{(12)} \rangle]^2} \left. \right\} \cdot \frac{1}{\Delta_2} \dot{u}_{\alpha}^{(2)} + \frac{u_{\alpha}^{(2)}}{\Delta_1^3} \langle u^{(2)}, \dot{u}^{(2)} \rangle = \frac{Q_1}{c^2} \left[ \frac{c^2 - \langle u^{(1)}, u^{(2)} \rangle \xi_{\alpha}^{(21)} - [c^2 \tau_{12} - \langle u^{(1)}, \xi^{(21)} \rangle] u_{\alpha}^{(1)}}{[c^2 \tau_{12} - \langle u^{(1)}, \xi^{(12)} \rangle]^3} \times \Delta_{21}^4 + D_{21} \Delta_{21}^2 \langle \xi^{(21)}, \dot{u}^{(1)} \rangle + \left( \langle \xi^{(21)}, u^{(1)} \rangle - c^2 \tau_{21} \right) \langle u^{(1)}, \dot{u}^{(1)} \rangle \right] + D_{21} \frac{\left( \langle u^{(2)}, \xi^{(21)} \rangle - c^2 \tau_{21} \right) \dot{u}_{\alpha}^{(1)} - \langle u^{(2)}, \dot{u}^{(1)} \rangle \xi_{\alpha}^{21}}{[c^2 \tau_{21} - \langle u^{(1)}, \xi^{(21)} \rangle]^2} + \frac{D_{21}}{\Delta_{21}^2} \cdot \frac{\left( \langle u^{(2)}, \xi^{(21)} \rangle - c^2 \tau_{21} \right) u_{\alpha}^{(1)} \langle u^{(1)}, \dot{u}^{(1)} \rangle}{[c^2 \tau_{21} - \langle u^{(1)}, \xi^{(21)} \rangle]^2} + \frac{D_{21}}{\Delta_{21}^2} \cdot \frac{\left( c^2 - \langle u^{(2)}, u^{(1)} \rangle \right) \xi_{\alpha}^{(21)} \langle u^{(1)}, \dot{u}^{(1)} \rangle}{[c^2 \tau_{21} - \langle u^{(1)}, \xi^{(21)} \rangle]^2} \left. \right\} \cdot \quad (3_{2\alpha})$$

Recall that in the above equations (3<sub>1α</sub>)

$$u^{(1)} = u^{(1)}(t), u^{(2)} = u^{(2)}(t - \tau_{12})$$

while in (3<sub>2α</sub>)

$$u^{(2)} = u^{(2)}(t), u^{(1)} = u^{(1)}(t - \tau_{21}).$$

We note that the delay functions  $\tau_{pq}(t)$  satisfy functional equations (\*) for  $t \in (-\infty, \infty)$ . For  $t \leq 0$   $u_\alpha^{(p)}(t)$  are prescribed functions:  $u_\alpha^{(p)}(t) = \bar{u}_\alpha^{(p)}(t), t \leq 0$ , where  $\bar{u}_\alpha^{(p)}(t) = \frac{d\bar{x}_\alpha^{(p)}(t)}{dt}, t \leq 0$ .

This means that for prescribed trajectories  $(\bar{x}_1^{(1)}(t), \bar{x}_2^{(1)}(t), \bar{x}_3^{(1)}(t)), (\bar{x}_1^{(2)}(t), \bar{x}_2^{(2)}(t), \bar{x}_3^{(2)}(t))$  for  $t \leq 0$  one has to find trajectories, satisfying the above system of equations for  $t > 0$ .

(We recall  $x_\alpha^{(p)}(t) = x_{\alpha 0}^{(p)} + \int_0^t u_\alpha^{(p)}(s) ds$ , where  $x_{\alpha 0}^{(p)}$  are the coordinates of the initial positions.)

**Kepler problem in polar coordinates**

In what follows we consider plane motion in  $Ox_2x_3$  coordinate plane for above equations. We suppose that the first particle  $P_1$  is fixed at the origin  $O(0,0,0)$ , that is,

$$P_1 : \begin{cases} x_1^{(1)}(t) = 0 \\ x_2^{(1)}(t) = 0, \quad t \in (-\infty, \infty) \\ x_3^{(1)}(t) = 0 \end{cases}$$

It follows by necessity  $\begin{cases} \bar{x}_1^{(1)}(t) = 0 \\ \bar{x}_2^{(1)}(t) = 0 \\ \bar{x}_3^{(1)}(t) = 0 \end{cases}$

Passing to the polar coordinates we can put

$$P_1 : \begin{cases} x_1^{(2)}(t) = 0 \\ x_2^{(2)}(t) = \rho(t) \cos \varphi(t) \\ x_3^{(2)}(t) = \rho(t) \sin \varphi(t) \end{cases}, \text{ where } \rho(t) > 0.$$

After transformations made in (Angelov, 2000) we obtain the following second order system:

$$\ddot{\rho}(t) = \rho(t)\dot{\varphi}^2(t) + \frac{Q_2}{c^3} \frac{\left[ c^2 - \dot{\rho}^2(t) \right] \sqrt{c^2 - \dot{\rho}^2(t) - \rho^2(t)\dot{\varphi}^2(t)}}{\rho^2(t)} \quad (4)$$

$$\ddot{\varphi}(t) = - \frac{2\dot{\rho}(t)\dot{\varphi}(t)}{\rho(t)} \left[ 1 + \frac{Q_2 \sqrt{c^2 - \dot{\rho}^2(t) - \rho^2(t)\dot{\varphi}^2(t)}}{2c^3 \rho(t)} \right]$$

for  $t > 0$  and initial conditions

$$\rho(0) = \rho_0, \dot{\rho}(0) = \dot{\rho}_0, \varphi(0) = \varphi_0, \dot{\varphi}(0) = \dot{\varphi}_0.$$

On the other hand beginning with the original form of Synge equations (Angelov, 2002) we obtain for Kepler problem the following equations of motion:

$$\frac{d(\gamma_2 u_\alpha^{(2)})}{dt} = \frac{Q_2 \xi_\alpha^{(21)}}{\rho^3} \quad (\alpha = 1,2,3). \quad (5_\alpha)$$

But  $\xi^{(21)} = (0, \rho(t) \cos \varphi(t), \rho(t) \sin \varphi(t))$ . Then integrating (5<sub>α</sub>) from 0 to t we have

$$\gamma_2(t) u_\alpha^{(2)}(t) - \gamma_2^0 u_\alpha^{(2)}(0) = Q_2 \int_0^t \frac{\xi_\alpha^{(21)}(s)}{\rho^3(s)} ds \text{ or}$$

$$\begin{cases} \dot{\rho}(t) \cos \varphi(t) - \rho(t) \dot{\varphi}(t) \sin \varphi(t) = \frac{1}{\gamma_2(t)} \left[ Q_2 \int_0^t \frac{\cos \varphi(s)}{\rho^2(s)} ds + C_2 \right] \\ \dot{\rho}(t) \sin \varphi(t) + \rho(t) \dot{\varphi}(t) \cos \varphi(t) = \frac{1}{\gamma_2(t)} \left[ Q_2 \int_0^t \frac{\sin \varphi(s)}{\rho^2(s)} ds + C_3 \right] \end{cases} \quad (6)$$

where  $C_2 = \gamma_2^0 (\dot{\rho}_0 \cos \varphi_0 - \rho_0 \dot{\varphi}_0 \sin \varphi_0)$ ,  
 $C_3 = \gamma_2^0 (\dot{\rho}_0 \sin \varphi_0 + \rho_0 \dot{\varphi}_0 \cos \varphi_0)$ ,  
 $\gamma_2^0 = \frac{1}{\sqrt{1 - \frac{1}{c^2} (\dot{\rho}_0^2 + \rho_0^2 \dot{\varphi}_0^2)}}$ .

The systems (6) and (5<sub>α</sub>) are equivalent. Indeed, the right-hand side of (5<sub>α</sub>) is the vector-function  $\vec{F}(t) = \left( \frac{\cos \varphi(t)}{\rho^2(t)}, \frac{\sin \varphi(t)}{\rho^2(t)} \right)$  and  $|\vec{F}(t) - \vec{F}(t_0)|^2 = \frac{1}{\rho^4(t)} + \frac{1}{\rho^4(t_0)} - \frac{2 \cos(\varphi(t) - \varphi(t_0))}{\rho^2(t)\rho^2(t_0)} \xrightarrow{t \rightarrow t_0} 0$ , that is,  $\vec{F}(t)$  is continuous vector-function – the necessary and sufficiently condition for equivalence of (6) and (5<sub>α</sub>).

On the other hand  $\dot{\gamma}_2 = \frac{d}{dt} \left( \frac{c}{\Delta_2} \right) = \frac{c}{\Delta_2^3} \langle u, \dot{u} \rangle$ ,

$$\frac{d}{dt} (\gamma_2 u_\alpha^{(2)}) = \dot{\gamma}_2 u_\alpha^{(2)} + \gamma_2 \dot{u}_\alpha^{(2)}, \quad (\alpha = 2,3)$$

Then from (5<sub>α</sub>) we obtain the system (with  $u \equiv u^{(2)}$ ):



$$\begin{cases} \frac{c}{\Delta_2^3} \langle u, \dot{u} \rangle u_2 + \frac{c}{\Delta_2} \dot{u}_2 = \frac{Q_2 \cos \varphi}{\rho^2} \\ \frac{c}{\Delta_2^3} \langle u, \dot{u} \rangle u_3 + \frac{c}{\Delta_2} \dot{u}_3 = \frac{Q_2 \sin \varphi}{\rho^2} \end{cases} \quad (7)$$

Multiplying the first equations of (7) by  $u_2$ , the second - by  $u_3$  and summing we obtain:

$$\frac{c}{\Delta_2^3} \langle u, \dot{u} \rangle (\langle u, u \rangle + \Delta_2^2) = \frac{Q_2}{\rho^2} (u_2 \cos \varphi + u_3 \sin \varphi), \text{ that}$$

$$\text{is } \frac{c}{\Delta_2^3} \langle u, \dot{u} \rangle = \frac{Q_2 \dot{\rho}}{c^2 \rho^2}.$$

$$\text{Thus we have } \dot{\gamma}_2 = \frac{Q_2 \dot{\rho}}{c^2 \rho^2}.$$

Multiplying the first equations of (7) by  $\cos \varphi$ , the second - by  $\sin \varphi$  and summing, and next multiplying the first equations of (7) by  $-\sin \varphi$ , the second - by  $\cos \varphi$  and

$$\text{summing, we obtain the system } \begin{cases} \dot{\gamma}_2 \dot{\rho} + \gamma_2 (\ddot{\rho} - \rho \dot{\varphi}^2) = \frac{Q_2}{\rho^2}, \\ \dot{\gamma}_2 \rho \dot{\varphi} + \gamma_2 (2\dot{\rho} \dot{\varphi} + \rho \ddot{\varphi}) = 0 \end{cases},$$

$$\text{or } (\dot{\gamma}_2 = \frac{Q_2 \dot{\rho}}{c^2 \rho^2})$$

$$\begin{cases} \gamma_2 (\ddot{\rho} - \rho \dot{\varphi}^2) = \frac{Q_2}{c^2 \rho^2} (c^2 - \dot{\rho}^2) \\ \gamma_2 (\rho \ddot{\varphi} + 2\dot{\rho} \dot{\varphi}) = - \frac{Q_2}{c^2 \rho} \dot{\rho} \dot{\varphi} \end{cases} \quad (8)$$

The final system (8) is equivalent to the system (4), since

$$\gamma_2 = \frac{c}{\Delta_2} = \frac{c}{\sqrt{c^2 - \dot{\rho}^2 - \rho^2 \dot{\varphi}^2}}.$$

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