ON THE EQUIVALENCE OF DIFFERENTIAL SYSTEMS ARISING IN ELECTROMAGNETIC TWO-BODY PROBLEM

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SUMMARY
The equivalence of two systems of equations of motion arising in electromagnetic two-body problem is obtained.

In the present note we consider two systems of equations of motion arising in electromagnetic two-body problem (Synge, 1940; Synge, 1960) and formulated in (Angelov, 2002).

First we recall some denotations and results from (Angelov, 2002; Angelov, 2000) concerning J. L. Synge's equations of motion

As in (Synge, 1940) we denote by

\[ x^{(p)}(t) = (x_1^{(p)}(t), x_2^{(p)}(t), x_3^{(p)}(t), x_4^{(p)}(t)) = itc \]

\( p = 1, 2 \), \( t^2 = -1 \) the space-time coordinates of the moving particles, by \( m_p \) - their proper masses, by \( e_p \) - their charges, \( c \) - the speed of the light. The coordinates of the velocity vectors are

\[ u^{(p)} = (u_1^{(p)}(t), u_2^{(p)}(t), u_3^{(p)}(t), u_4^{(p)}(t)) \quad (p = 1, 2) \].

The coordinates of the unit tangent vectors to the world-lines are

\[ \gamma^{(p)} = \frac{\gamma^{(p)}(t)}{c} = \frac{u^{(p)}(t)}{c} \quad (\alpha = 1, 2, 3), \quad \gamma^{(p)} = itc \quad (p = 1, 2) \].

where

\[ \gamma_p = \left(1 - \left(\sum_{\alpha=1}^{3} u^{(p)}_{\alpha}(t) \right)^2 \right)^{-\frac{1}{2}}, \quad \Delta_p = \left(\sum_{\alpha=1}^{3} u^{(p)}_{\alpha}(t) \right)^2 \].

It follows \( \gamma_p = c / \Delta_p \).

By \( < , >_4 \) we denote the scalar product in the Minkowski space, while by \( < , >_3 \) - the scalar product in 3-dimensional Euclidean subspace. Synge's equations of motion modeling the interaction of two moving charged particles are the following:

\[ m_p \frac{dx^{(p)}(t)}{ds_p} = \frac{e_p}{c^2} F_{(p)}^{(p)} \delta^{(p)}(r) \quad (r = 1, 2, 3, 4) \quad (1) \]

where the elements of proper time are

\[ ds_p = \frac{c}{\gamma_p} \Delta_p dt = \Delta_p dt \quad (p = 1, 2) \].

Recall that in (1) there is a summation in \( n \) \( (n = 1, 2, 3) \).

The elements \( F_{(p)}^{(p)} \) of the electromagnetic tensors are derived by the retarded Lienard-Wiecerd potentials

\[ A_\alpha^{(p)} = -\frac{e_p}{\gamma^{(p)}(t)} \frac{\delta^{(p)}(r)}{\xi^{(p)q}} \quad (r = 1, 2, 3, 4) \], that is

\[ F_{(p)}^{(p)} = \frac{\partial A^{(p)}_{\alpha}}{\partial x^{(p)}_\alpha} - \frac{\partial A^{(p)}_{\alpha}}{\partial x^{(p)}_\beta} \xi^{(pq)} \]

By \( \xi^{(pq)} \) we denote the isotropic vectors (cf. Synge, 1940; 1960) drawn into the past:

\[ \xi^{(pq)} = (x_1^{(p)}(t) - x_1^{(q)}(t - \tau_{pq}(t)), \quad x_2^{(p)}(t) - x_2^{(q)}(t - \tau_{pq}(t)), x_3^{(p)}(t) - x_3^{(q)}(t - \tau_{pq}(t)), x_4^{(p)}(t) - x_4^{(q)}(t - \tau_{pq}(t)), itc \tau_{pq}(t)) \]
where \( \xi^{(p,q)}(\xi^{(p,q)}) \) for \( p \) or
\[
\tau_{pq}(t) = \left( \frac{1}{\gamma_p} \sum_{\beta=1}^{d} \left[ \chi^{(p)}_\beta(t) - \chi^{(q)}_\beta(t - \tau_{pq}(t)) \right]^2 \right)^{\frac{1}{2}}
\]
\( ((pq) = (12), (21)) \).

Calculating \( F^{(p)}_n \) as in (Angelov, 1990) we write equations from (2) in the form:
\[
d\alpha(p)_d = \frac{\partial (\xi^{(p)}(\alpha^{(p)}))}{\partial s_d} \left[ \xi^{(p)}(\xi^{(p)}) \right]_{d} - \xi^{(q)}(\xi^{(p)}) \left[ \xi^{(p)}(\xi^{(p)}) \right]_{d} \frac{1}{\gamma_d} + \left[ \xi^{(p)}(\xi^{(p)}) \right]_{d} \frac{\partial (\xi^{(p)}(\xi^{(p)}))}{\partial q_d} \left[ \xi^{(p)}(\xi^{(p)}) \right] d_q - \left[ \xi^{(p)}(\xi^{(p)}) \right]_{d} \frac{\partial (\xi^{(p)}(\xi^{(p)}))}{\partial q_d} \left[ \xi^{(p)}(\xi^{(p)}) \right]_{d}
\]
\( \alpha = 1, 2, 3 \) (2.2)

\[
d\alpha(p)_d = \frac{\partial (\xi^{(p)}(\alpha^{(p)}))}{\partial s_d} \left[ \xi^{(p)}(\alpha^{(p)}) \right]_{d} - \xi^{(q)}(\alpha^{(p)}) \left[ \xi^{(p)}(\alpha^{(p)}) \right]_{d} \frac{1}{\gamma_d} + \left[ \xi^{(p)}(\alpha^{(p)}) \right]_{d} \frac{\partial (\xi^{(p)}(\alpha^{(p)}))}{\partial q_d} \left[ \xi^{(p)}(\alpha^{(p)}) \right] d_q - \left[ \xi^{(p)}(\alpha^{(p)}) \right]_{d} \frac{\partial (\xi^{(p)}(\alpha^{(p)}))}{\partial q_d} \left[ \xi^{(p)}(\alpha^{(p)}) \right]_{d}
\]
\( \alpha = 1, 2, 3 \) (3.1a)

where \( \tau_{pq}(t) = \left( \frac{1}{\gamma_p} \sum_{\beta=1}^{d} \left[ \chi^{(p)}_\beta(t) - \chi^{(q)}_\beta(t - \tau_{pq}(t)) \right]^2 \right)^{\frac{1}{2}} \)

\[
\frac{d\alpha(p)}{ds_d} = \frac{\partial (\xi^{(p)}(\alpha^{(p)}))}{\partial s_d} \left[ \xi^{(p)}(\alpha^{(p)}) \right]_{d} - \frac{1}{\gamma_d} + \left[ \xi^{(p)}(\alpha^{(p)}) \right]_{d} \frac{\partial (\xi^{(p)}(\alpha^{(p)}))}{\partial q_d} \left[ \xi^{(p)}(\alpha^{(p)}) \right] d_q - \left[ \xi^{(p)}(\alpha^{(p)}) \right]_{d} \frac{\partial (\xi^{(p)}(\alpha^{(p)}))}{\partial q_d} \left[ \xi^{(p)}(\alpha^{(p)}) \right]_{d}
\]
\( \alpha = 1, 2, 3 \) (3.1a)

where \( \gamma_{pq} = \frac{1}{c^2} \sum_{\alpha=1}^{d} \left[ \xi^{(q)}_{\alpha}(t - \tau_{pq}(t)) \right]^2 \),
\( \Delta_{pq} = \left[ c^2 - \sum_{\alpha=1}^{d} \xi^{(q)}_{\alpha}(t - \tau_{pq}(t)) \right]^2 \) and
\[
\left[ \xi^{(p)}(\alpha^{(p)}) \right]_{d} \frac{\partial (\xi^{(p)}(\alpha^{(p)}))}{\partial q_d} \left[ \xi^{(p)}(\alpha^{(p)}) \right]_{d} + \frac{1}{\gamma_d} \left[ \xi^{(p)}(\alpha^{(p)}) \right]_{d} \frac{\partial (\xi^{(p)}(\alpha^{(p)}))}{\partial q_d} \left[ \xi^{(p)}(\alpha^{(p)}) \right]_{d}
\]
\( \alpha = 1, 2, 3 \) (3.1a)

where \( \tau_{pq}(t) = \left( \frac{1}{\gamma_p} \sum_{\beta=1}^{d} \left[ \chi^{(p)}_\beta(t) - \chi^{(q)}_\beta(t - \tau_{pq}(t)) \right]^2 \right)^{\frac{1}{2}} \)

\[
\frac{d\alpha(p)}{ds_d} = \frac{\partial (\xi^{(p)}(\alpha^{(p)}))}{\partial s_d} \left[ \xi^{(p)}(\alpha^{(p)}) \right]_{d} - \frac{1}{\gamma_d} + \left[ \xi^{(p)}(\alpha^{(p)}) \right]_{d} \frac{\partial (\xi^{(p)}(\alpha^{(p)}))}{\partial q_d} \left[ \xi^{(p)}(\alpha^{(p)}) \right] d_q - \left[ \xi^{(p)}(\alpha^{(p)}) \right]_{d} \frac{\partial (\xi^{(p)}(\alpha^{(p)}))}{\partial q_d} \left[ \xi^{(p)}(\alpha^{(p)}) \right]_{d}
\]
\( \alpha = 1, 2, 3 \) (3.1a)

where \( \gamma_{pq} = \frac{1}{c^2} \sum_{\alpha=1}^{d} \left[ \xi^{(q)}_{\alpha}(t - \tau_{pq}(t)) \right]^2 \),
\( \Delta_{pq} = \left[ c^2 - \sum_{\alpha=1}^{d} \xi^{(q)}_{\alpha}(t - \tau_{pq}(t)) \right]^2 \) and
\[
\left[ \xi^{(p)}(\alpha^{(p)}) \right]_{d} \frac{\partial (\xi^{(p)}(\alpha^{(p)}))}{\partial q_d} \left[ \xi^{(p)}(\alpha^{(p)}) \right]_{d} + \frac{1}{\gamma_d} \left[ \xi^{(p)}(\alpha^{(p)}) \right]_{d} \frac{\partial (\xi^{(p)}(\alpha^{(p)}))}{\partial q_d} \left[ \xi^{(p)}(\alpha^{(p)}) \right]_{d}
\]
\( \alpha = 1, 2, 3 \) (3.1a)
\[ \frac{1}{A_2} \Delta_2 \mathbf{u}(t_2) + \int_0^{t_2} \Delta_2(t) \mathbf{u}(t) dt = \mathbf{q}(t_2) \]

\[ \Delta_2 = \frac{\mathbf{u}(t_2) - \mathbf{u}(t_1)}{t_2 - t_1} \]

We note that the delay functions \( \tau_{pq}(t) \) satisfy functional equations \((*)\) for \( t \in (-\infty, \infty) \). For \( t \leq 0 \) \( u_{\alpha}^{(p)}(t) \) are prescribed functions: \( u_{\alpha}^{(p)}(t) = \pi_{\alpha}^{(p)}(t), t \leq 0 \), where \( \pi_{\alpha}^{(p)}(t) = \frac{dx_{\alpha}^{(p)}(t)}{dt} \), \( t \leq 0 \).

This means that for prescribed trajectories \( (x_1^{(1)}(t), x_2^{(1)}(t), x_2^{(2)}(t), x_3^{(1)}(t), x_3^{(2)}(t), x_3^{(3)}(t)) \) for \( t \leq 0 \) one has to find trajectories, satisfying the above system of equations for \( t > 0 \).

We recall \( x_{\alpha}^{(p)}(t) = x_{\alpha}^{(p)}(0) + \int_0^t u_{\alpha}^{(p)}(s) ds \), where \( x_{\alpha}^{(p)}(0) \) are the coordinates of the initial positions.)

Kepler problem in polar coordinates

In what follows we consider plane motion in \( Ox_2x_3 \) coordinate plane for above equations. We suppose that the first particle \( P_1 \) is fixed at the origin \( O(0,0,0) \), that is,

\[ P_1: \begin{cases} x_1^{(1)}(t) = 0, \\ x_2^{(1)}(t) = 0, \quad t \in (-\infty, \infty), \\ x_3^{(1)}(t) = 0 \end{cases} \]

It follows by necessity \( x_2^{(2)}(t) = 0 \), \( x_3^{(2)}(t) = 0 \).

Passing to the polar coordinates we can put

\[ P_1: \begin{cases} x_1^{(2)}(t) = \rho(t) \cos \phi(t), \\ x_2^{(2)}(t) = \rho(t) \sin \phi(t) \end{cases} \]

After transformations made in (Angelov, 2000) we obtain the following second order system:

\[ \frac{d}{dt} \left( \begin{array}{c} \xi^{(1)} \\ \xi^{(2)} \end{array} \right) = \left( \begin{array}{c} \rho(t) \xi^{(1)}(t) \\ \rho(t) \xi^{(2)}(t) \end{array} \right) - \left( \begin{array}{c} \xi^{(1)}_0 \\ \xi^{(2)}_0 \end{array} \right) \]

\[ \left( \begin{array}{c} \xi^{(1)}_0 \\ \xi^{(2)}_0 \end{array} \right) = \left( \begin{array}{c} \rho_0 \cos \phi_0 \\ \rho_0 \sin \phi_0 \end{array} \right) \]

for \( t > 0 \) and initial conditions

\[ \rho(0) = \rho_0, \phi(0) = \phi_0 \]

On the other hand, beginning with the original form of Syngue equations (Angelov, 2002) we obtain for Kepler problem the following equations of motion:

\[ \frac{d}{dt} \left( \begin{array}{c} \gamma_2^{(1)}(t) \\ \gamma_2^{(2)}(t) \end{array} \right) = \left( \begin{array}{c} \rho_0 \gamma_2^{(1)}(t) \\ \rho_0 \gamma_2^{(2)}(t) \end{array} \right) \]

\[ \left( \begin{array}{c} \rho_0 \gamma_2^{(1)}(t) \\ \rho_0 \gamma_2^{(2)}(t) \end{array} \right) = \left( \begin{array}{c} \rho_0 \cos \phi_0 \\ \rho_0 \sin \phi_0 \end{array} \right) \]

But \( \xi^{(2)}_0 = (0, \rho(t) \cos \phi(t), \rho(t) \sin \phi(t)) \). Then integrating \( (5) \) from 0 to \( t \) we have

\[ \gamma_2^{(1)}(t)u_\alpha^{(2)}(t) - \gamma_2^{(2)}u_\alpha^{(2)}(0) = \frac{\rho_0}{2} \int_0^t \left( \begin{array}{c} \frac{\xi^{(2)}_0}{\rho_0} \cos \phi_0 - \rho_0 \phi_0 \sin \phi_0 \\ \phi_0 - \cos \phi_0 \end{array} \right) ds \]

\[ \left( \begin{array}{c} \frac{\rho_0}{2} \cos \phi_0 - \rho_0 \phi_0 \sin \phi_0 \\ \phi_0 - \cos \phi_0 \end{array} \right) \]

\[ \left( \begin{array}{c} \frac{\rho_0}{2} \cos \phi_0 - \rho_0 \phi_0 \sin \phi_0 \\ \phi_0 - \cos \phi_0 \end{array} \right) \]

\[ \left( \begin{array}{c} \rho_0 \cos \phi_0 - \rho_0 \phi_0 \sin \phi_0 \\ \phi_0 - \cos \phi_0 \end{array} \right) \]

\[ \gamma_2^{(1)} = \frac{\rho_0}{2} \left( \begin{array}{c} \cos \phi_0 - \rho_0 \phi_0 \sin \phi_0 \\ \phi_0 - \cos \phi_0 \end{array} \right) \]

\[ \gamma_2^{(1)}(t)u_\alpha^{(2)}(t) - \gamma_2^{(2)}u_\alpha^{(2)}(0) = \frac{\rho_0}{2} \int_0^t \left( \begin{array}{c} \frac{\xi^{(2)}_0}{\rho_0} \cos \phi_0 - \rho_0 \phi_0 \sin \phi_0 \\ \phi_0 - \cos \phi_0 \end{array} \right) ds \]

\[ \left( \begin{array}{c} \frac{\rho_0}{2} \cos \phi_0 - \rho_0 \phi_0 \sin \phi_0 \\ \phi_0 - \cos \phi_0 \end{array} \right) \]

\[ \left( \begin{array}{c} \rho_0 \cos \phi_0 - \rho_0 \phi_0 \sin \phi_0 \\ \phi_0 - \cos \phi_0 \end{array} \right) \]

\[ \left( \begin{array}{c} \rho_0 \cos \phi_0 - \rho_0 \phi_0 \sin \phi_0 \\ \phi_0 - \cos \phi_0 \end{array} \right) \]

\[ \left( \begin{array}{c} \rho_0 \cos \phi_0 - \rho_0 \phi_0 \sin \phi_0 \\ \phi_0 - \cos \phi_0 \end{array} \right) \]
The systems (6) and \((S_\alpha)\) are equivalent.

Indeed, the right-hand side of \((S_\alpha)\) is the vector-function

\[
P(t) = \left( \frac{\cos \varphi(t)}{\rho^2(t)}, \frac{\sin \varphi(t)}{\rho^2(t)} \right)
\]

and

\[
\frac{d^2}{dt^2} \left( \frac{\cos \varphi(t)}{\rho^2(t)} \right) - \frac{1}{\rho^4(t)} \left( \frac{\cos \varphi(t) - \varphi(t_0)}{\rho^2(t) \rho^2(t_0)} \right) \to 0 \quad \text{as} \quad t \to t_0,
\]

that is, \(P(t)\) is continuous vector-function – the necessary and sufficiently condition for equivalence of (6) and \((S_\alpha)\).

On the other hand \(\dot{\psi}_2 = \frac{d}{dt} \left( \frac{c}{\Delta_2} \right) \cdot \frac{c}{\Delta_2} \{u, \psi_2\},\)

\[
\frac{d}{dt} \left( \psi_2 \left( \begin{array}{c}
\alpha \\
\alpha
\end{array} \right) \right) = \psi_2 \psi_2^{(2)}(\alpha) + \gamma_2 \psi_2^{(2)}(\alpha), \quad (\alpha = 2, 3)
\]

Then from \((S_\alpha)\) we obtain the system (with \(u = u^{(2)}\)):

\[
\frac{c}{\Delta_2} \{u, \psi_2\} \psi_2 + \frac{c}{\Delta_2} \psi_2 = \frac{O_2 \cos \varphi}{\rho^2},
\]

\[
\frac{c}{\Delta_2} \{u, \psi_3\} \psi_3 + \frac{c}{\Delta_2} \psi_3 = \frac{O_2 \sin \varphi}{\rho^2}.
\]

Multiplying the first equations of (7) by \(u_2\), the second - by \(u_3\) and summing we obtain:

\[
\frac{c}{\Delta_2} \{u, \psi_2\} \{u, u\} + \frac{c}{\Delta_2} \psi_2 = \frac{O_2 \cos \varphi + u_3 \sin \varphi}{\rho^2},
\]

that is

\[
\frac{c}{\Delta_2} \{u, \psi_2\} = \frac{O_2 \psi_2}{c^2 \rho^2}.
\]

Thus we have \(\psi_2 = \frac{O_2 \psi_2}{c^2 \rho^2}\).

Multiplying the first equations of (7) by \(\cos \varphi\), the second – by \(\sin \varphi\) and summing, and next multiplying the first equations of (7) by \(\sin \varphi\), the second - by \(\cos \varphi\) and summing, we obtain the system

\[
\begin{align*}
\psi_2 \psi_2^{(2)}(\alpha) (\rho \psi_2^2) + \gamma_2 (\rho \psi_2^2) = \frac{O_2}{\rho^2}, & \quad \text{or} \\
\psi_2 \psi_2^{(2)}(\alpha) (\rho \psi_2^2) + \gamma_2 (2\rho \psi_2^2) = 0
\end{align*}
\]

\[
(\psi_2 = \frac{O_2 \psi_2}{c^2 \rho^2})
\]

\[
\psi_2 \left( \begin{array}{c}
\rho \psi_2^2 \\
\rho \psi_2^2
\end{array} \right) = \frac{O_2}{c^2 \rho^2} \left( c^2 - \rho^2 \right)
\]

\[
\psi_2 \left( \begin{array}{c}
\rho \psi_2^2 \\
\rho \psi_2^2
\end{array} \right) = -\frac{O_2}{c^2 \rho^2} \rho \psi_2 \\
\psi_2 \left( \begin{array}{c}
\rho \psi_2^2 \\
\rho \psi_2^2
\end{array} \right) = 0
\]

(8)

The final system (8) is equivalent to the system (4), since

\[
\psi_2 = \frac{\rho \psi_2}{c^2 \rho^2}.
\]

REFERENCES


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